The South African Mathematical Olympiad
Third Round 2006
Senior Division (Grades 10 to 12)
Solutions

1.

$$\begin{aligned}\frac{2121212121210}{1121212121211} &= \frac{2121212121210 \div 3}{1121212121211 \div 3} \\ &= \frac{707070707070}{373737373737} \\ &= \frac{70 \times 10101010101}{37 \times 10101010101} \\ &= \frac{70}{37}\end{aligned}$$

2. B is on the circle with midpoint C and radius 1. Then $\widehat{A}$ reaches its maximum value if AB is tangent to the circle ($\sin\widehat{A} = \dfrac{\sin\widehat{B}}{2}$ reaches a maximum if $\sin\widehat{B}$ reaches a maximum, i.e., if $\widehat{B} = 90°$.) In this case, $\sin\widehat{A} = 1/2$, giving $\widehat{A} = 30°$.

3. Note that $14^2 = 196$. Suppose that the square of $n = m + 14$ also ends in 196, for some integer m. Then $1000 \mid n^2 - 14^2$, i.e., $1000 \mid m(m+28)$, and it follows that m is a multiple of 4, otherwise it is either odd (which contradicts $1000 \mid m(m+28)$), or it is of the form $4a+2$ (which forces $250 \mid (2a+1)(2a+15)$, another contradiction). Say $m = 4b$. Then $250 \mid 2b(b+7)$, so that $125 \mid b(b+7)$. Since b and $b+7$ do not have 5 as a common factor, we have two possibilities:

   (a) $b = 125k$ for some $k \in \mathbb{Z}$. Then $m = 500k$, i.e., $n = 500k + 14$.

   (b) $b + 7 = 125k$ for some $k \in \mathbb{Z}$. Then $m = 500k - 28$, i.e., $n = 500k - 14$.

   It is easily checked that $(500k \pm 14)^2 \equiv 196 \pmod{1000}$ for all $k \in \mathbb{Z}$. The positive integers, the squares of which end in 196, is therefore given by the set $\{14\} \cup \{500k \pm 14 : k \in \mathbb{Z}, k > 0\}$.

4. *Solution I (Geometrical)*: Construct DE, with E on BC, such that $BE = BD$. Construct DF, with F on AB, such that $FD \| BC$. Since $B\widehat{D}E = B\widehat{E}D = 80°$, we have $C\widehat{E}D = 100°$ and $C\widehat{D}E = 40°$. Furthermore, $A\widehat{F}D = A\widehat{D}F = 40°$ and

$BF = DC$ are consequences of $FD \| BC$. It follows that $F\widehat{D}B = 20^\circ = F\widehat{B}D$, and we see that $BF = FD = DC$. But then triangles AFD and ECD are congruent ($FD = CD$, and all angles coincide), implying that $AD = EC$. Consequently, $BC = BE + EC = BD + AD$.

Solution II (Trigonometrical) by Poobhalan Pillay: Assume, without loss of generality, that $AB = AC = 1$. Then $BC = 2\cos 40^\circ$. By the sine rule in triangle ABD,

$$AD = \frac{\sin 20^\circ}{\sin 60^\circ} \quad \text{and} \quad BD = \frac{\sin 100^\circ}{\sin 60^\circ},$$

giving

$$AD + BD = \frac{\sin(60^\circ - 40^\circ) + \sin(60^\circ + 40^\circ)}{\sin 60^\circ} = \frac{2\sin 60^\circ \cos 40^\circ}{\sin 60^\circ} = BC.$$

5. *Solution I (by Dirk Laurie)*: Every allowable k-element subset corresponds to a way of choosing k out of a row of 10 objects so that no two are adjacent, e.g.

$$\circ \ \circ \ \bullet \ \circ \ \bullet \ \circ \ \circ \ \bullet \ \circ \ \circ$$

Remove $k-1$ unselected objects, one form each gap, e.g.

$$\circ \ \circ \ \bullet \ \bullet \ \circ \ \bullet \ \circ \ \circ$$

This establishes, for each $k \geqslant 2$, a one-to-one correspondence between allowable subsets of $\{1, 2, \dots, 10\}$ containing k elements, and the number of ways of choosing k out of $10 - k + 1$ objects. It follows that there are

$$\binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5} = 36 + 56 + 35 + 6 = 133$$

allowable subsets.

Solution II (by Johan Meyer): Let s_n be the number of allowable subsets of $\{1, 2, \dots, n\}$. Clearly $s_1 = s_2 = 0$. Suppose S is an allowable subset of $\{1, 2, \dots, n+1\}$, and let $T = S \cap \{1, 2, \dots, n\}$. If $n + 1 \notin S$, then $T = S$. If $n + 1 \in S$, then either T is an allowable subset of $\{1, 2, \dots, n-1\}$ or $T = \{k\}$, with $k \in \{1, 2, \dots, n-1\}$. Hence

$$s_{n+1} = s_n + s_{n-1} + n - 1.$$

This gives $s_3 = 1, s_4 = 3, s_5 = 7, s_6 = 14, s_7 = 26, s_8 = 46, s_9 = 79, s_{10} = 133$.

6. *Solution I (by Johan Meyer)*: To solve (a), it is sufficient to show that the sequence $f(1), f(2), f(3), \ldots$ contains a strictly increasing subsequence. We find (using $\lfloor x \rfloor + \lfloor y \rfloor \leqslant \lfloor x + y \rfloor$) that:

$$\begin{aligned}
f(2n) &= \tfrac{1}{2n}\left(\left\lfloor \tfrac{2n}{1} \right\rfloor + \left\lfloor \tfrac{2n}{2} \right\rfloor + \left\lfloor \tfrac{2n}{3} \right\rfloor + \cdots + \left\lfloor \tfrac{2n}{2n} \right\rfloor\right) \\
&\geqslant \tfrac{1}{2n}\left(2\left\lfloor \tfrac{n}{1} \right\rfloor + 2\left\lfloor \tfrac{n}{2} \right\rfloor + \cdots + 2\left\lfloor \tfrac{n}{n} \right\rfloor + \left\lfloor \tfrac{2n}{n+1} \right\rfloor + \cdots + \left\lfloor \tfrac{2n}{2n} \right\rfloor\right) \\
&> \tfrac{1}{2n}\left(2\left\lfloor \tfrac{n}{1} \right\rfloor + 2\left\lfloor \tfrac{n}{2} \right\rfloor + \cdots + 2\left\lfloor \tfrac{n}{n} \right\rfloor\right) \\
&= \tfrac{1}{n}\left(\left\lfloor \tfrac{n}{1} \right\rfloor + \left\lfloor \tfrac{n}{2} \right\rfloor + \cdots + \left\lfloor \tfrac{n}{n} \right\rfloor\right) \\
&= f(n).
\end{aligned}$$

This gives a strictly increasing subsequence $f(1) < f(2) < f(4) < f(8) < \cdots$.

To solve (b), we note that if $p \geqslant 3$ is prime and $2 \leqslant k \leqslant p-1$, then the integral parts of $\frac{p}{k}$ and $\frac{p-1}{k}$ coincide, i.e., $\lfloor \frac{p}{k} \rfloor = \lfloor \frac{p-1}{k} \rfloor$. Let $S_p = \sum_{k=2}^{p-1} \lfloor \frac{p}{k} \rfloor = \sum_{k=2}^{p-1} \lfloor \frac{p-1}{k} \rfloor$. Then $f(p) = \frac{1}{p}(p + S_p + 1) = 1 + \frac{1}{p}(S_p + 1)$ and $f(p-1) = 1 + \frac{1}{p-1} S_p$, so that $f(p-1) - f(p) > 0$ if and only if $S_p > p - 1$. But if we choose $p \geqslant 7$, then $S_p > p-1$, since, in this case, $\frac{p-1}{2} \geqslant 3$, $\frac{p-1}{3} \geqslant 2$, and $\frac{p-1}{k} \geqslant 1$ for $4 \leqslant k \leqslant p-1$. As there are infinitely many primes, (b) is solved.

Solution II (by Dirk Laurie): Let $g(n) = nf(n)$. Note that $\lfloor \frac{n}{k} \rfloor - \lfloor \frac{n-1}{k} \rfloor = 0$ except when k is a divisor of n, in which case $\lfloor \frac{n}{k} \rfloor - \lfloor \frac{n-1}{k} \rfloor = 1$. It follows that $g(n) = g(n-1) + d(n)$, where $d(n)$ is the number of positive divisors of n, giving $f(n) = (d(1) + d(2) + \cdots + d(n))/n$, That is to say, $f(n)$ equals the arithmetic mean of $d(1), d(2), \ldots, d(n)$. Thus, it is sufficient to prove that $d(n+1) > f(n)$ infinitely often, and $d(n+1) < f(n)$ infinitely often.

Now $d(1) = 1$, and when $n > 1, d(n) \geqslant 2$, with equality if and only if n is prime. Since $f(6) = \frac{7}{3}$, it follows that $f(n) > 2$ for all $n \geqslant 6$.

(a) Since $d(2^k) = k + 1$, the sequence $d(1), d(2), d(3), \ldots$ is unbounded, and it happens infinitely often that $d(n+1) > \max\{d(1), d(2), \ldots, d(n)\}$. For all such n, $d(n+1) > f(n)$.

(b) Since there are infinitely many primes, it happens infinitely often that $d(n+1) = 2$. For all such n, $d(n+1) < f(n)$.