The South African Mathematics Olympiad
Third Round 2011
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Consider the sequence $2, 3, 5, 6, 7, 8, 10, \ldots$ of all positive integers that are not perfect squares. Determine the 2011^{th} term of this sequence.

2. Suppose that x and y are real numbers that satisfy the system of equations

$$2^x - 2^y = 1,$$
$$4^x - 4^y = \tfrac{5}{3}.$$

   Determine $x - y$.

3. We call a sequence of m consecutive integers a *friendly* sequence if its first term is divisible by 1, the second by 2, ..., the $(m-1)^{\text{th}}$ by $m-1$, and in addition, the last term is divisible by m^2. Does a friendly sequence exist for (a) $m = 20$ (b) $m = 11$?

4. An airline company is planning to introduce a network of connections between the ten different airports of Sawubonia. The airports are ranked by priority from first to last (with no ties). We call such a network *feasible* if it satisfies the following conditions:

   - All connections operate in both directions.
   - If there is a direct connection between two airports A and B, and C has a higher priority than B, then there must also be a direct connection between A and C.

   Some of the airports may not be served, and even the empty network (no connections at all) is allowed. How many feasible networks are there?

5. Let $\mathbb{N}_0$ denote the set of all nonnegative integers. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ with the following two properties:

   (a) $0 \le f(x) \le x^2$ for all $x \in \mathbb{N}_0$;

   (b) $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}_0$ with $x > y$.

6. In triangle ABC, the incircle touches BC in D, CA in E and AB in F. The bisector of $\angle BAC$ intersects BC in G. The lines BE and CF intersect in J. The line through J perpendicular to EF intersects BC in K. Prove that

$$\frac{GK}{DK} = \frac{AE}{CE} + \frac{AF}{BF}.$$