THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior Second Round 2011

Solutions

1. **Answer A.** The first number is 1 plus no 3's, the second number is 1 plus one 3, the third number is 1 plus two 3's, and so on. In this way, it follows that the 100th number is 1 plus 99 3's, which is $1 + 99 \times 3 = 298$. [The numbers are terms in an arithmetic sequence. A standard formula for the nth term of an arithmetic sequence with first term a and common difference d is $a + (n - 1)d$. In this case, $a = 1$ and $d = 3$ and $n = 100$.]

2. **Answer A.** There are two paths from A to the middle point and two paths from the middle point to B. These can be combined to make a total of $2 \times 2 = 4$ paths from A to B. [If U denotes an upward move along one side of a square and R denotes a move to the right, then the four paths can be labelled RURU, RUUR, URRU, and URUR.]

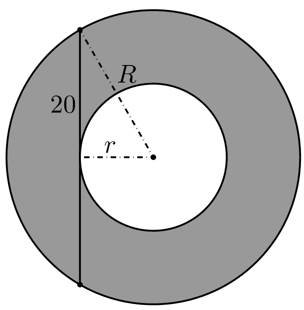
3. **Answer B.** We need four different positive integers whose sum is 15. If the largest one is to be as small as possible, then the smaller ones must be as large as possible, so we should try to get successive numbers. If the largest number is n, then we need $n + (n - 1) + (n - 2) + (n - 3) \geq 15$, so $4n - 6 \geq 15$, giving $n \geq 5\frac{1}{4}$. Since n is a whole number, the smallest value of n is 6. Alternatively, by trial and error, we try $4 + 3 + 2 + 1 = 10$, which is too small, and $5 + 4 + 3 + 2 = 14$, which is still too small. We then see that $6+4+3+2$ works, and no smaller value than 6 is possible.

4. **Answer B.** To obtain the largest value of $\frac{M+N}{M-N}$ we must make $M + N$ as large as possible, and $M - N$ as small as possible, but still positive. The smallest positive value of $M - N$ is 1, in which case the largest value of $M + N$ occurs when $M = 50$ and $N = 49$. The fraction is then $\frac{50+49}{50-49} = 99$.

5. **Answer C.** One can of orange paint contains $\frac{3}{5}$ of a can of red paint and $\frac{2}{5}$ of a can of yellow paint. One can of green paint contains $\frac{2}{3}$ of a can of blue paint and $\frac{1}{3}$ of a can of yellow paint. When orange and green are mixed, we have $(\frac{2}{5} + \frac{1}{3})$ of a can of yellow paint in a total of 2 cans, so the proportion is $(\frac{2}{5} + \frac{1}{3}) \div 2 = \frac{11}{30}$.

6. **Answer C.** We need to know how many sets of 13 people (12 guests and 1 waiter) can be formed from 400 people. Now $400 = 30 \times 13 + 10$. But those last 10 people in the remainder also need a waiter. Therefore 31 waiters are required, which means $400 - 31 = 369$ guests can be served.

7. **Answer E.** Use units of cm, and let the radii of the circles be R and r, with $R > r$. In the diagram, we apply Pythagoras' theorem to obtain $R^2 - r^2 = 20^2 = 400$. The area of the shaded region between the two circles is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2) = 400\,\pi$. Alternatively, since the radii of the circles are not given, one can let r shrink to zero, leaving the vertical line segment as the diameter of the large circle and $R = 20$. The area is therefore $\pi\, 20^2 = 400\,\pi$.



8. **Answer A.** The number 10 000 $= 10^4 = 2^4 5^4$ must be written as the product of two factors with no zeros among their digits. If either factor is divisible by both 2 and 5, then its last digit will be zero, which is not allowed. Therefore the only possible factors are $2^4 = 16$ and $5^4 = 625$, whose sum is 641.

9. **Answer B.** We draw up a table of the possible sums that can be obtained with a throw of these two dice. (The first row gives the first die, and the first column gives the second die.)

	1	1	2	3	5	8
1	2	2	3	4	6	9
1	2	2	3	4	6	9
2	3	3	4	5	7	10
3	4	4	5	6	8	11
5	6	6	7	8	10	13
8	9	9	10	11	13	16

Among the 36 sums in the table there are 14 Fibonacci numbers (circled), so the probability is $\frac{14}{36} = \frac{7}{18}$.

10. **Answer D.** Pair off the factors 17 and 23, then the 18 and 22, and then the 19 and 21, leaving the middle factor 20 on its own. Now use the fact that $(20-n)(20+n) = 20^2 - n^2$, which is approximately 20^2 when $n = 1$, 2, or 3. The product of the three pairs is therefore approximately 20^6, and the extra 20 in the middle makes $20^7 = 2^7 \times 10^7 = 1.28 \times 10^9$. (The actual value of the product is 1.24×10^9, when rounded to three significant digits.)

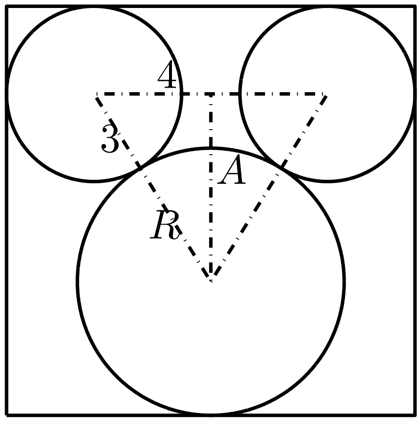
11. **Answer D.** Suppose the area of the circle is C and the area of the square is S. Since the region between the circle and the square consists of four flaps, it follows that $C - S = 4F$, where F denotes the area of a flap. From the diagram it is clear that the shaded region is obtained by removing four flaps from the square, so its area is $S - 4F = S - (C - S) = 2S - C$. The circle has radius 1, so $C = \pi 1^2 = \pi$. Since the diagonal of the square is 2 (diameter of the circle), its sides are of length $\sqrt{2}$ by Pythagoras' theorem, so $S = (\sqrt{2})^2 = 2$. The required area is therefore $2S - C = 2 \times 2 - \pi = 4 - \pi$.

12. **Answer B.** We are given $2011 = M^2 - N^2 = (M - N)(M + N)$. Therefore, both $M - N$ and $M + N$ are factors of 2011. The simplest factorization to try is 1×2011, that is, $M - N = 1$ and $M + N = 2011$. Solving these two equations simultaneously gives $M = 1006$ and $N = 1005$. That 1×2011 is the only possible factorization follows from the fact that 2011 is a prime number. (If this is news to you, convince yourself of this fact by dividing 2011 by all prime numbers up to 43.)

13. **Answer E.** We need to know how many times 3 appears as a factor in $100! = 1 \times 2 \times 3 \times \cdots \times 99 \times 100$. In the product, every number divisible by 3 provides a factor 3, and there are 33 of these $(3, 6, 9, 12, \ldots, 99)$. Furthermore, every number divisible by 3^2 provides an additional factor 3, and there are 11 of these $(9, 18, \ldots, 99)$. Similarly, every number divisible by 3^3 provides yet another factor 3, and there are 3 of these

$(27, 54, 81)$. Finally, the number $3^4 = 81$ provides one last factor 3. Therefore the total number of times 3 divides 100! is $33 + 11 + 3 + 1 = 48$.

14. **Answer E.** Suppose the large circle has radius R. The centres of the circles form an isosceles triangle with two sides of length $R+3$ and one side of length 8. Let A be the altitude of this triangle. By Pythagoras' theorem, $A^2 = (R+3)^2 - 4^2 = R^2 + 6R - 7$. Next, the side of the square is equal to the altitude plus the sum of the radii of the circles, so $14 = A + R + 3$. This gives $A = 11 - R$ and $A^2 = 121 - 22R + R^2$. By equating the expressions for A^2 we see that $R^2 + 6R - 7 = 121 - 22R + R^2$, which simplifies to $28R = 128$, so $R = \frac{32}{7}$.



15. **Answer A.** Let $t = 3^x$. Then $3^x - 9^x = t - t^2$, and since x can be any real number, it follows that t can be any positive real number. Thus we need to find the maximum value of $t - t^2$ for all positive values of t. Now $t - t^2 = \frac{1}{4} - (t - \frac{1}{2})^2$ by completing the square. Since $(t - \frac{1}{2})^2 \geq 0$, it follows that the maximum value of $t - t^2$ is $\frac{1}{4}$ and occurs when $t = \frac{1}{2}$, which is positive, as required.

16. **Answer D.** Label the numbers A, B, C, D, E as in the answers. We write each number in the form $1/(\sqrt{x} + \sqrt{y})$, and then compare the x's and y's. To do this for A we multiply (and divide) by its conjugate

$$A = \sqrt{11} - \sqrt{10} = \left(\sqrt{11} - \sqrt{10}\right)\frac{\sqrt{11} + \sqrt{10}}{\sqrt{11} + \sqrt{10}} = \frac{11 - 10}{\sqrt{11} + \sqrt{10}} = \frac{1}{\sqrt{11} + \sqrt{10}}.$$

In exactly the same way $\quad B = \sqrt{10} - 3 = \sqrt{10} - \sqrt{9} = \dfrac{1}{\sqrt{10} + \sqrt{9}}.$

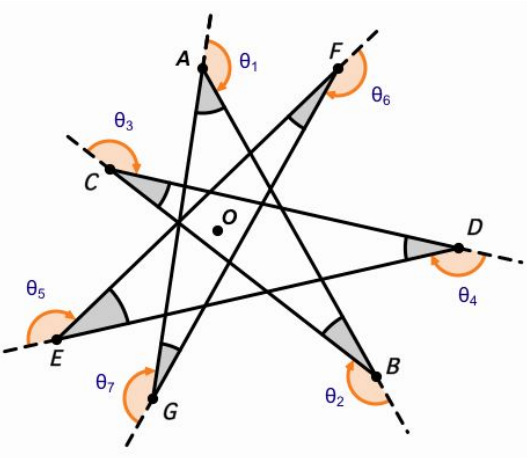
For C and D,

$$C = \frac{1}{6} = \frac{1}{\sqrt{9} + \sqrt{9}}, \quad D = \frac{\sqrt{10}}{20} = \frac{\sqrt{10} \times \sqrt{10}}{20 \times \sqrt{10}} = \frac{1}{2\sqrt{10}} = \frac{1}{\sqrt{10} + \sqrt{10}}.$$

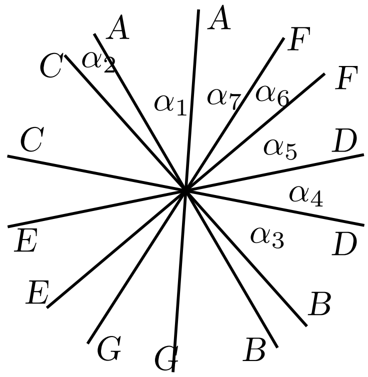
E follows in the same way as D, namely $E = \dfrac{\sqrt{11}}{22} = \dfrac{1}{\sqrt{11} + \sqrt{11}}.$

By comparing denominators we conclude $E < A < D < B < C$.

17. **Answer D.** *Solution 1:* Start at any vertex and follow along all the lines until you have returned to the starting point and turned to face your original direction. The interior angles at the vertices are given by $180° - \theta_i$, where $i = 1, 2, \ldots, 7$. The total sum of the interior angles is therefore $7 \times 180° - \theta$, where $\theta = \theta_1 + \theta_2 + \cdots + \theta_7$. Now let O be any point in the middle region of the diagram. During your walk you have gone three times around O. (Imagine a vertical rod fixed at O with a string joining it to you. As you do your walk, the string will wind three times around the rod.) Since you have encircled O three times, you have turned through a total of 3×360 degrees, in other words, $\theta = 3 \times 360°$. Therefore, the sum of the interior angles is $7 \times 180° - 3 \times 360° = 180°$. [The proof can be adapted to show that the sum of the exterior angles of a convex polygon is 360°. Likewise, one can show that the sum of the interior angles of a convex n-gon is $(n-2)180°$. In both these cases you encircle the point O only once.]



Solution 2: Label the shaded interior angles $\angle A = \alpha_1$, $\angle B = \alpha_2$, etc. Now translate every line parallel to itself so that all seven lines meet in a single point, as shown in the diagram. By marking the location of each α in the diagram, we see that the seven angles complete half a revolution.



18. **Answer C.** *Solution 1:* When considering relative motion, it is often helpful to bring things to rest. So imagine an observer sitting in a boat floating with the goggles (that is, at rest relative to the river). This observer will see James swim away for 10 minutes, and return in another 10 minutes, since relative to the observer James swims the same distance at the same speed in both directions. The observer floats 500 metres in 20 minutes, at speed $\frac{1}{2}$km/$\frac{1}{6}$h, or 1.5 km/h.

Solution 2: For a more brute force approach, imagine the river is a number line and the direction of flow is positive. Let J be James's speed in the water, and let R be the speed of the river. We work in kilometres and hours, as in the answers. When James swims upstream, his speed relative to the ground is $J - R$ and when he swims downstream his speed is $J + R$. If James is at the origin when he loses his goggles, swimming in a negative direction, then after $\frac{1}{6}$ h he is at the point $-\frac{1}{6}(J - R)$ and the goggles have floated to the point $\frac{1}{6}R$. He then turns round and catches his goggles at the point $\frac{1}{2}$, which means he has swum a distance of $(\frac{1}{2} + \frac{1}{6}(J - R))$ km at a speed of $(J+R)$ km/h, so the time taken is $(\frac{1}{2}+\frac{1}{6}(J-R))/(J+R)$ h. In the same time, the goggles have floated a distance of $(\frac{1}{2}-\frac{1}{6}R)$ km at a speed of R km/h, so the time taken is $(\frac{1}{2}-\frac{1}{6}R)/R$. Since the times are equal, we can equate them and cross-multiply to get $R(\frac{1}{2}+\frac{1}{6}(J-R)) = (J+R)(\frac{1}{2}-\frac{1}{6}R)$. This simplifies to $JR = \frac{3}{2}J$, and we can cancel J (which is not zero!) to get $R = \frac{3}{2}$. (Note that James' speed is undetermined, aside from the fact that $J > \frac{3}{2}$.)

19. **Answer B.** There is exactly one point available for each game, so since the winner has $4\frac{1}{2}$ points, she (or he) must have played at least five rounds. Thus there must have been at least six contestants. Assuming for the moment there were exactly six contestants, then there were 15 games. (Each player plays five games, but each game involves two players, so the number of games is $\frac{1}{2}(6\times 5)$.) Hence there was a total of 15 points available, and the top four players have a total of $4\frac{1}{2}+3\frac{1}{2}+3+1\frac{1}{2} = 12\frac{1}{2}$. The bottom two players therefore had a total of $2\frac{1}{2}$ points, and neither of them got more than $1\frac{1}{2}$ points, or they wouldn't have been at the bottom. The only way to divide $2\frac{1}{2}$ points between them is $1\frac{1}{2}+1$, so the loser got 1 point. To see that there could not have been more than six contestants, suppose there were n contestants, where $n \geq 6$. The average score of the bottom $n-4$ contestants is $\frac{1}{2}(n(n-1)-25)/(n-4)$, which cannot be more than $\frac{3}{2}$. This gives $n(n-1)-25 \leq 3(n-4)$ which simplifies to $n \leq 2+\sqrt{17} \approx 6.1$, so the only possibility is $n = 6$.

20. **Answer D.** No one shakes more than six hands, and my seven friends all do different numbers of handshakes, so these numbers can only be 0,1,2,3,4,5,6. We label my seven friends by their numbers of handshakes. Friend 6 has shaken hands with everyone, which means everyone should have a count of at least 1. The exception is the partner of friend 6, who should therefore be friend 0. Now friend 5 shook hands with friend 6, as well as with four other people. Since these four people shook hands with both friends 5 and 6, none of them can be friend 1. Therefore friends 1 and 5 are partners, and therefore friend 5 shook hands with friends 6, 2, 3, 4 and with me. Friend 4 shook hands with friend 5 and 6, as well as with two other persons. Since these two people shook hands with friends 6, 5 and 4, none of them can be friend 2. Therefore friends 2 and 4 are partners, and therefore friend 4 shook hands with friends 6, 5, 3 and with me. It follows that friend 3 is my partner and that we both shook hands with friends 4, 5, and 6; hence I shook hands with three people. [The reader is encouraged to draw a graph, showing the connections at each step of the argument.]