THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior Second Round 2010

Solutions

1. **Answer C.**
On the first round, numbers 2, 4, 6, 8, 10 go away, leaving 1, 3, 5, 7, 9 at the table. At the next round, numbers 3 and 7 go away, leaving 1, 5, 9 behind. Finally, numbers 1 and 9 go away, and number 5 remains.

2. **Answer C.**
Suppose that x learners belong to both clubs. Then $15 - x$ belong to science only, and $12 - x$ belong to chess only. Therefore $(15 - x) + (12 - x) = 13$, so $27 - 2x = 13$, giving $2x = 14$ and $x = 7$.

3. **Answer E.**
Let x be the number on the second card. Then the number on the third card is $20 - 2 - x = 18 - x$, the number on the fourth card is $20 - x - (18 - x) = 2$ again, and the number on the fifth card is $20 - (18 - x) - 2 = x$ again. Now the pattern repeats in cycles of length three: $2, x, 18–x, 2, x, 18–x, \ldots$. It follows that the number on the ninth card is the same as the number on the third card, so $8 = 18 - x$, giving $x = 10$.

4. **Answer A.**
The wedge has two square faces with sides of length 1, two triangular faces with hypotenuse of length $\sqrt{2}$, and one rectangular face with sides of length 1 and $\sqrt{2}$. The total surface area is thus $2 \times 1 + 2 \times \frac{1}{2} + \sqrt{2} = 3 + \sqrt{2}$.

5. **Answer B.**
Starting inside the bracket, we have $(3 \uparrow) \times (5 \downarrow) = (1 - 3) \times (1 + 5) = (-2) \times 6 = -12$. The full expression is therefore $(-12) \uparrow = 1 - (-12) = 13$.

6. **Answer A.**
The sum of the n numbers is mn. If we add 5 to each number and then multiply by k, then the total becomes $k(mn + 5n)$ and the mean is $k(m + 5)$.

7. **Answer C.**
Each power of 100 has two digits fewer than the next higher power, so there is no carrying. Thus the number of digits in the sum is the same as the number of digits in 100^{200}, which equals 10^{400} and has 401 digits. (Remember that 10^n has $n + 1$ digits.)

8. **Answer A.**
The fractions have a common denominator 180, and their sum is

$$\frac{60 + 30 + 20 + 15 + 12 + 10}{180} = \frac{147}{180} = \frac{120 + 27}{180} = \frac{2}{3} + \frac{1}{12} + \frac{1}{15}.$$

The fractions removed are $\frac{1}{12}$ and $\frac{1}{15}$, and their product is $\frac{1}{180}$.

9. **Answer B.**
Let M be the point of intersection of the diagonals. Then, using Pythagoras' theorem, we have $MA^2 + MB^2 = 9$ and $MA^2 + MD^2 = 16$ and $MD^2 + MC^2 = 36$. Finally,

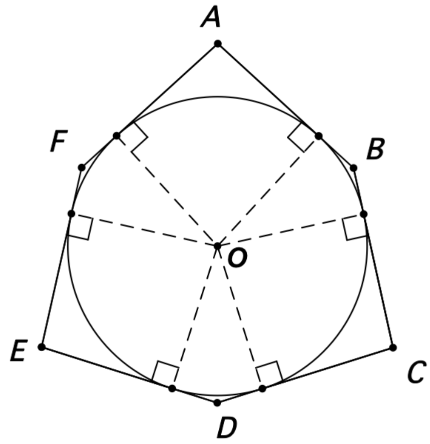
$$BC^2 = MB^2 + MC^2 = (MA^2 + MB^2) - (MA^2 + MD^2) + (MD^2 + MC^2) = 9 - 16 + 36,$$

so $BC = \sqrt{29}$.

10. **Answer D.**
Since x is positive, we have $0 < 1 - 2^{-x} < 1$, and $1 - 2^{-x} \approx 1$ when x is large. It follows that $x - 1 \approx 2010$, so $x \approx 2011$.

11. **Answer E.**

Join the six points of tangency to the centre O of the circle. (Note that since the hexagon is not regular, these points are not the midpoints of the sides.) This divides the hexagon into six kites, each of which is made up of two congruent right-angled triangles. Because the sides of the hexagon are of equal length, it follows that each kite is not congruent to the two kites next to it, but is congruent to the two kites after those. Thus the kites at vertices A, C, E are congruent to one another, and so are the kites at vertices B, D, F. The angle sum of a hexagon equals $2 \times 6 - 4$ right-angles, which is $720°$. Thus $720° = \widehat{A} + \widehat{B} + \widehat{C} + \widehat{D} + \widehat{E} + \widehat{F} = 3\widehat{A} + 3\widehat{D}$, giving $\widehat{D} = (720 - 3 \times 130)/3 = 110°$.



12. **Answer B.**
The radius from the centre $(a; b)$ to the point of tangency $(b; a)$ has slope $\frac{b-a}{a-b} = -1$. The tangent is perpendicular to the radius, so it has slope $-1/(-1) = 1$. (Remember the slopes of two perpendicular lines have product -1.)

13. **Answer B.**
Substituting $x = 2$ gives $f(2) + f(\frac{1}{2}) = 1$ and substituting $x = \frac{1}{2}$ gives $-\frac{1}{2}f(\frac{1}{2}) + f(2) = 1$. The first equation plus twice the second equation simplifies to $3f(2) = 3$, so $f(2) = 1$.

14. **Answer A.**
Each of the triangles has angles $30°, 60°, 90°$, so its shortest side is half its hypotenuse. The shortest side CD of triangle BCD is of length $\frac{1}{2}BC = \frac{1}{2}$. In addition, CD is the hypotenuse of the next triangle CDE, so $DE = \frac{1}{2}CD = \frac{1}{4}$, and so on. Therefore $CD + DE + EF + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots = (\frac{1}{2})/(1 - \frac{1}{2}) = 1$, using the formula $a/(1-r)$ for the sum of an infinite geometric series.

15. **Answer D.**
Solution 1: For $n > 5$ we have $1 < \frac{n+3}{n-1} < 2$ and for $n < -3$ we have $0 < \frac{n+3}{n-1} < 1$, so we need to consider the nine integer values of n from -3 to $+5$ only.

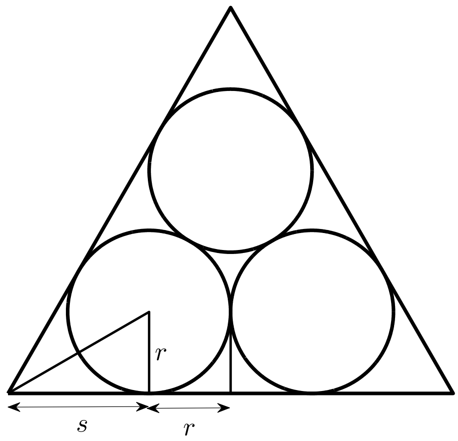
n	-3	-2	-1	0	1	2	3	4	5
$\frac{n+3}{n-1}$	0	$-\frac{1}{3}$	-1	-3	undefined	5	3	$\frac{7}{3}$	2

We find that $\frac{n+3}{n-1}$ is an integer except for $n = -2$, $n = 1$, and $n = 4$, giving six acceptable values of n.

Solution 2: Write $\dfrac{n+3}{n-1} = \dfrac{n-1+4}{n-1} = 1 + \dfrac{4}{n-1}$. The number on the far left is an integer if and only if $4/(n-1)$ is an integer. Since the divisors of 4 are 1, 2 and 4, the only possibilities are $n-1=\pm1$, $n-1=\pm2$, and $n-1=\pm4$, leading to $n=0,2,-1,3,-3,5$, as listed in the table above.

16. **Answer B.**
There are four colours available for M, but only three colours for P, which is adjacent to M, and only two colours for U, which is adjacent to both M and P (of different colours). There are again three colours available for A, which is adjacent to M, and also three colours for L and three for G, which are both adjacent to A. Finally, there are only two colours available for G, which is adjacent to both A and N (different). This gives a total of $4\times3\times2\times3\times3\times3\times2=1296$ colourings.

17. **Answer E.**



In the little triangle we have

$$\frac{r}{s}=\tan 30^\circ=\frac{1}{\sqrt{3}} \quad\Longrightarrow\quad s=r\sqrt{3}$$

Therefore

$$s+r=\frac{1}{2} \quad\Longrightarrow\quad r(\sqrt{3}+1)=\frac{1}{2}$$

and so

$$r=\frac{1}{2+2\sqrt{3}}.$$

18. **Answer C.**
The sum of all ten numbers is $1+2+\cdots+10=55$. The sum of eight of the numbers lies between $1+2+\cdots+8=36$ and $3+4+\cdots+10=52$. If this sum is to be a multiple of four the only possibilities are $36, 40, 44, 48, 52$. Hence the sum of the chosen two numbers must be one of $19, 15, 11, 7, 3$. Let us count the number of ways that these numbers can be obtained as the sum of two numbers between 1 and 10. There is exactly one way to obtain 3 $(1+2)$, three ways to obtain 7 $(1+6,\ 2+5,\ 3+4)$, five ways to obtain 11 $(1+10,\ \ldots,\ 5+6)$, three ways to obtain 15 $(5+10,\ 6+9,\ 7+8)$ and one way to obtain 19 $(9+10)$. This gives a total of 13 possible choices. Finally, there are 45 ways to select two numbers from a set of 10 (ten possible choices for the first one, and nine for the second giving 90 possibilities, but since each pair is counted twice we have to divide by two). The probability is therefore $\frac{13}{45}$.

19. **Answer E.**
Since the ratio of the areas of the triangles is $4:3$, it follows that the ratio of their sides is $\sqrt{4}:\sqrt{3}=2:\sqrt{3}$. Thus $AC:XZ=2:\sqrt{3}$ so $XZ=\sqrt{3}$. The three triangles ZAX, XBY, YCZ are all congruent to one another, so if $AZ=x$, then $AX=ZC=2-x$. Now by the Cosine Rule $XZ^2=AX^2+AZ^2-2AX.AZ\cos 60^\circ$, so $3=(2-x)^2+x^2-(2-x)x$ which simplifies to $3x^2-6x+1=0$. The solution is $x=(3\pm\sqrt{6})/3$, and we choose the minus sign for AZ, because $AZ<\frac{1}{2}AC=1$. (The other solution gives the length of ZC.)

20. **Answer D.**
Solution 1: Note that the numbers 5, 7, 9, 11 satisfy the conditions, except for the fact that their mutual differences are all 2 (rather than 1, as required). Hence we would like to divide by 2, which is not yet possible since the numbers are all odd. So we first add an odd number to all

of them, which must be a multiple of 5, 7, 9 and 11. The number $5 \times 7 \times 9 \times 11 = 3465$ satisfies this condition. We obtain the numbers 3470, 3472, 3474 and 3476, which are multiples of 5, 7, 9 and 11 respectively. Dividing by 2, we find consecutive numbers with this property: 1735, 1736, 1737 and 1738.

Solution 2: Let the smallest of the four numbers be $N = 5k$ where k is a positive integer. Then 7 divides into $N + 1 = 5k + 1 = 7k - (2k - 1)$. Therefore $(2k - 1)$ is divisible by 7. Similarly 9 divides into $N + 2 = 5k + 2 = 9k - 2(2k - 1)$. Therefore $(2k - 1)$ is also divisible by 9. And $N + 3 = 5k + 3 = 11k - 3(2k - 1)$ shows that $(2k - 1)$ is also divisible by 11. Therefore $(2k - 1)$ is divisible by $7 \times 9 \times 11 = 693$, and if we put $2k - 1 = 693$ we get that $N = 5k = 1735$.
