THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior Second Round 2009

Solutions

1. **Answer C.** The numbers are $n-2,\ n-1,\ n,\ n+1,\ n+2$, with sum $5n$.

2. **Answer E.** The area of a triangle is half the base times the height, while the area of a parallelogram is base times height. Thus the parallelogram has twice the area of the triangle, that is $2 \times 14 = 28$.

3. **Answer C.** Put $k = \dfrac{a}{2} = \dfrac{b}{3} = \dfrac{c}{5}$; then $810 = abc = (2k)(3k)(5k) = 30k^3$. Thus $k^3 = 810/30 = 27$, so $k = 3$ and $b = 3k = 9$.

4. **Answer A.** Nico's total for eight tests was $8 \times 85 = 680$ and for nine tests was $9 \times 81 = 729$. Therefore his mark in the ninth test was $729 - 680 = 49$.

5. **Answer D.** $\langle 1 \rangle = 1$ (given). Next, $\langle 2 \rangle = \langle 1+1 \rangle = \langle 1 \rangle + \langle 1 \rangle + 1 \times 1 = 1+1+1 = 3$. Finally, $\langle 3 \rangle = \langle 1+2 \rangle = \langle 1 \rangle + \langle 2 \rangle + 1 \times 2 = 1+3+2 = 6$.

6. **Answer B.** First method: the bottom-left L-shaped portion of the square consists of one shaded square and two white squares, so $\frac{1}{3}$ of it is shaded. The rest of the square is filled up with smaller copies of this, which are all similar to one another, and therefore have the same fraction that is shaded. Thus $\frac{1}{3}$ of the whole square will eventually be shaded.
   Second method: the largest shaded square has area $\frac{1}{4}$ of the whole square. The next one is half its size, therefore quarter its area, so it is $\frac{1}{16}$ of the whole square. Thus the total shaded area, as a fraction of the whole square, is
   $$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}.$$

7. **Answer D.** There are 36 possible ways of throwing a dice twice. If the first throw is 1, then there are five possibilities for a greater throw in the second; if the first throw is 2, then there are four, and so on. Thus the total number where the second throw is greater than the first is $5+4+3+2+1+0 = 15$, so the required probability is $\dfrac{15}{36} = \dfrac{5}{12}$.

8. **Answer C.** Suppose the radius of the circle is r. Then the points of tangency divide the shortest side into segments with lengths r and $5-r$, and the next side into segments with lengths r and $12-r$. The hypotenuse is divided into segments with lengths $12-r$ and $5-r$, so $(12-r)+(5-r) = 13$, giving $r = 2$.

9. **Answer D.** By Pythagoras' theorem, the bases of the triangles are $1, \sqrt{2}, \sqrt{3}, \ldots$, so the base of the 100th triangle is $\sqrt{100} = 10$. Since all triangles have height 1, the area of the 100th triangle is $\frac{1}{2} \times 10 \times 1 = 5$.

10. **Answer A.** Imagine a second semicircle drawn on side XY, giving another region congruent to A. Then we see that if we remove the large semicircle from the triangle and the two small semicircles, then we are left with two copies of A. Thus $2A = \frac{1}{2} \times 1 \times 1 + \frac{1}{2}\pi(\frac{1}{2})^2 + \frac{1}{2}\pi(\frac{1}{2})^2 - \frac{1}{2}\pi(\frac{\sqrt{2}}{2})^2 = \frac{1}{2} + \frac{\pi}{4} - \frac{\pi}{4} = \frac{1}{2}$, so $A = \frac{1}{4}$, which is half the area of the triangle.
[The moon-shaped region A is called a *lune*, and the result is called Hippocrates' theorem on lunes. It says that the sum of the areas of the lunes on any right-angled triangle is equal to the area of the triangle. Try to prove it in general.]

11. **Answer B.** The general term is $\dfrac{1}{\sqrt{n} + \sqrt{n+1}}$, which can be simplified by multiplying above and below by the conjugate surd $\sqrt{n} - \sqrt{n+1}$, when it reduces to $\dfrac{\sqrt{n} - \sqrt{n+1}}{n - (n+1)} = -\sqrt{n} + \sqrt{n+1}$. Thus the expression can be rewritten as

$$(-\sqrt{1} + \sqrt{2}) + (-\sqrt{2} + \sqrt{3}) + (-\sqrt{3} + \sqrt{4}) + \cdots + (-\sqrt{2008} + \sqrt{2009}).$$

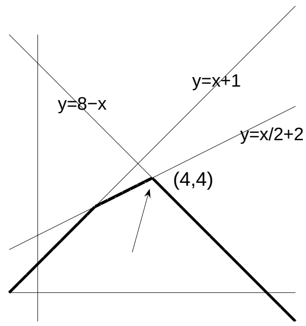
By rearranging the brackets we see that almost everything cancels, and we are left with $-\sqrt{1} + \sqrt{2009}$.

12. **Answer B.** With B as the origin $(0;0)$, suppose the points have the following co-ordinates: $C(c;0)$, $A(0;a)$, $P(x;y)$, $D(c;a)$. Then by Pythagoras' theorem $BP^2 = 16 = x^2 + y^2$ and similarly $AP^2 = 9 = x^2 + (a-y)^2$ and $CP^2 = 25 = (c-x)^2 + y^2$. Then $DP^2 = (c-x)^2 + (a-y)^2 = ((c-x)^2 + y^2) + (x^2 + (a-y)^2) - (x^2 + y^2) = 25 + 9 - 16 = 18$, so $DP = \sqrt{18} = 3\sqrt{2}$.
Notice that this proves that $AP^2 + PC^2 = BP^2 + PD^2$ for any point P inside rectangle $ABCD$. The co-ordinates are not essential: the result can be obtained by simply dropping perpendiculars from P to the four sides of the rectangle.

13. **Answer B.** The expressions $x + 1$ and $\dfrac{x}{2} + 2$ increase as x increases, but $8 - x$ decreases. The greatest value of $m(x)$ must occur near the points where the graphs cross, which are at $x = 3,5$ and $x = 4$ and $x = 2$. Try a few values to be sure:

x	$x+1$	$\frac{1}{2}x+2$	$-x+8$	$m(x)$
2	3	3	6	3
3	4	3,5	5	3,5
3,5	4,5	3,75	4,5	3,75
4	5	4	4	4
5	6	4,5	3	3

You can also draw rough graphs on the same axes if you like. At each x the graph of $m(x)$ is the lowest graph of the three. This is shown by the thicker curve in the figure, and the largest value is indicated by the arrow.



14. **Answer D.** This is an arithmetic sequence with 81 terms, and middle term $n+60$, so the sum is equal to $81(n+60)$ (as in Question 1). Since $81 = 9^2$, which is already a perfect square, we need the smallest value of n such that $n+60$ is a perfect square. By inspection, we see that $n = 4$, since $64 = 8^2$.

15. **Answer D.** By completing the square, we have $x^2 - 2xy + 2y^2 - 6y = (x-y)^2 + (y-3)^2 - 9$, which has a minimum value of -9 at $x = y = 3$, since the perfect squares are non-negative.

16. **Answer E.**

$$\frac{5}{9} \cdot \frac{12}{16} \cdot \frac{21}{25} \cdot \frac{32}{36} \cdots = \frac{1 \times 5}{3 \times 3} \cdot \frac{2 \times 6}{4 \times 4} \cdot \frac{3 \times 7}{5 \times 5} \cdot \frac{4 \times 8}{6 \times 6} \cdots .$$

Everything eventually cancels except for 1×2 in the numerator and 3×4 in the denominator, so the final answer is $\dfrac{2}{12} = \dfrac{1}{6}$.

17. **Answer B.** $93! + 94! + 95! = 93!(1 + 94 + (94)(95)) = 93!(95)^2 = 1 \times 2 \times 3 \times \cdots \times 93 \times (5 \times 19)^2$. The largest prime factor of the product will be the largest prime less than or equal to 93, which is 89.

18. **Answer C.** We use the conjugate surd, as in Question 11.

$$\sqrt{n} - \sqrt{n-1} = (\sqrt{n} - \sqrt{n-1}) \cdot \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}} = \frac{n - (n-1)}{\sqrt{n} + \sqrt{n-1}} = \frac{1}{\sqrt{n} + \sqrt{n-1}}.$$

The inequality becomes $\dfrac{1}{\sqrt{n} + \sqrt{n-1}} < \dfrac{1}{100}$, which is the same as $\sqrt{n} + \sqrt{n-1} > 100$. Since $\sqrt{n}$ and $\sqrt{n-1}$ are almost equal, this will first occur when $\sqrt{n} \approx 50$, that is, when $n \approx 2500$. Now use inspection: if $n = 2500$, then $\sqrt{n} = 50$ and $\sqrt{n-1} < 50$, so $\sqrt{n} + \sqrt{n-1} < 100$. However, when $n = 2501$, we have $\sqrt{n} > 50$ and $\sqrt{n-1} = 50$, so $\sqrt{n} + \sqrt{n-1} > 100$, as required.

19. **Answer C.** We need to calculate x^3, where $x = 10^{10} - 1$, so let us start with $(10^n - 1)^3$ for smaller values of n. It is useful to remember that $(a-1)^3 = a^3 - 3a^2 + 3a - 1$.

n	$10^{3n} - 3 \times 10^{2n}$	$3 \times 10^n - 1$	$(10^n - 1)^3$	digit sum
1	700	29	729	18
2	970 000	299	970 299	36
3	997 000 000	2 999	997 002 999	54
4	999 700 000 000	29 999	999 700 029,999	72

In each row there is one more 9 at the beginning, one more 9 at the end, and one more 0 in the middle. Thus the digit sum increases by 18 at each stage, and in row n it is $18n$. For $n = 10$ the answer is 180.

20. **Answer B.** Every possible path uses exactly four downward line segments (one for each level). There are $2 \times 4 \times 4 \times 2 = 64$ possible ways to choose these downward segments. Each choice gives exactly one path from A to B, since there is only one way to link the downward line segments by horizontal ones (if necessary). Therefore, there are 64 such paths.