THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION
Sponsored by HARMONY GOLD MINING

SECOND ROUND 2006

SENIOR SECTION: GRADES 10, 11 AND 12

17 MAY 2006

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. C
2. A
3. E
4. A
5. C
6. D
7. E
8. B
9. B
10. C
11. A
12. A
13. E
14. D
15. A
16. D
17. D
18. B
19. B
20. C

Private Bag X173, Pretoria, 0001 TEL: (012) 392-9323
FAX: (012) 320-7803 E-mail: ellie@samf.ac.za
Organisations involved: AMESA, SA Mathematical Society, SA Akademie vir Wetenskap en Kuns.

SOLUTIONS

1. **Answer C.** The time in years is $7 \text{ km} \div 0.1 \text{ mm} = 7 \times 10^3 \text{ m} \div 0.1 \times 10^{-3} \text{ m} = 7 \times 10^3 \div 10^{-4} = 7 \times 10^7 = 70$ million.

2. **Answer A.** $120\% = \frac{6}{5}$ and $75\% = \frac{3}{4}$, so $\frac{6}{5}E = \frac{3}{4}J$ (where E = Ellie's weight and J = James' weight). Thus $\frac{E}{J} = \frac{3}{4} \div \frac{6}{5} = \frac{3}{4} \times \frac{5}{6} = \frac{5}{8}$.

3. **Answer E.** If a denotes the unknown area, then $\frac{6}{a} = \frac{16}{32} = \frac{1}{2}$, so $a = 6 \times \frac{2}{1} = 12$.

4. **Answer A.** If d is the diameter of each ball, then the length of the container is $3d$ and its circumference is πd. The ratio is therefore $3 : \pi \approx 3 : 3.14 \approx 1 : 1$.

5. **Answer C.** If the digits of the number are $ABCD$, then $A \times B \times C \times D = 75 = 3 \times 5 \times 5$. Since there is a fourth digit, there must be a fourth factor on the right hand side, which can only be 1. Therefore $A + B + C + D = 3 + 5 + 5 + 1$ (in some order) $= 14$.

6. **Answer D.** Since $b^2 = b + 1$, statement (D) can be simplified to $b^3 = 0$, which is false. The other statements can easily be verified.

7. **Answer E.** $(a+b)^2 = a^2 + 2ab + b^2$, and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Thus $a^3 + +b^3 = (a+b)^3 - (3a^2b + 3ab^2) = (a+b)^3 - 3ab(a+b) = (-3)^3 - 3 \times 4 \times (-3) = -27 + 36 = 9$.

8. **Answer B.** To move from view 1 to view 2, the cube must be rotated through 180° about the axis through the middle of face C, since both face Q and face A have disappeared from sight. Thus in view 2, face Q must be at the bottom, so the face opposite it is the one on top, which is B.

9. **Answer B.** Let $a = 2 + \sqrt{3}$ and $b = 2 - \sqrt{3}$. Then $ab = \sqrt{2^2 - 3} = 1$, and $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b = (2 + \sqrt{3}) - 2\sqrt{1} + (2 - \sqrt{3}) = 2$. Thus $\sqrt{a} - \sqrt{b} = \pm\sqrt{2}$. Since $a > b$, it follows that $\sqrt{a} - \sqrt{b}$ is positive, so $\sqrt{a} - \sqrt{b} = +\sqrt{2}$.

10. **Answer C.** If the angles are arranged in increasing order, and the difference between successive angles is say $d°$, then the angles (in degrees) are $75,\ 75+d,\ 75+2d,\ 75+3d$. Since the sum of the angles in any quadrilateral is 360°, it follows that $300 + 6d = 360$, so $d = 10$, and the second largest angle is 95°.

11. **Answer A.** By trial and error (starting, for example, with $40^2 = 1600$), it is easy to find that the only square between 1800 and 1900 is $1849 = 43^2$. Thus de Morgan was 43 in 1849, so he was born in 1806.

12. **Answer A.** Imagine the side of length 20 as the base of the triangle, and the other two sides, whose total length is 52, as defining the height of the triangle. The maximum area comes when the height is a maximum, which is when both sides are equal, of length 26. (It can be shown that the apex of the triangle lies on an ellipse with the other two vertices as the foci.) The triangle is then isosceles, and can be split into two right-angled triangles with base 10 and hypotenuse 26. By Pythagoras' theorem, the height is 24, so the area is $10 \times 24 = 240$.

13. **Answer E.** Suppose n is any integer, and x is any number such that $n \le x < n + \frac{1}{2}$, so $\lfloor x \rfloor = n$. Then $n < x + \frac{1}{2} < n + 1$, so $\lfloor x + \frac{1}{2} \rfloor = n$ also. This gives infinitely many solutions for x, even with a single value of n.

14. **Answer D.** To find the average (or arithmetic mean), we add the numbers together and divide the sum by 9, but in this case it is easier to divide each number by 9 first and add them afterwards. The average is equal to

$$1 + 11 + 111 + \cdots + 111111111 = 123456789.$$

15. **Answer A.** The northerly component (or y-value) of the endpoint is $80 - 20 + 5 - \frac{5}{4} + \cdots$, in which each term is $-\frac{1}{4}$ times the preceding one. This is a geometric series, and its sum to infinity is $80 \div (1 - (-\frac{1}{4})) = 80 \times \frac{4}{5} = 64$. Similarly, the easterly component, or x-value, of the endpoint is $40 \times \frac{4}{5} = 32$.

16. **Answer D.** If you draw lines joining the point inside the pentagon to the five vertices, then you divide the pentagon into five triangles. The five sides, which are all of length say s, form the bases of the triangles, and the sum of the heights of the triangles is given to be 16. Thus the sum of the areas of the triangles is $\frac{1}{2}s \times 16 = 8s$. But this sum is the area of the pentagon, which is 40, so $8s = 40$ and $s = 5$.

17. **Answer D.** The number of ways of choosing three of the nine points is $(9 \times 8 \times 7) \div 6 = 84$. (There are nine ways to choose the first point, eight ways to choose the second, and seven ways to choose the third; however, since each choice of three points can be ordered in six different ways that make no difference to the triangle, we must divide the number of ordered choices by 6.) Each of the 84 choices of three points will give the vertices of a triangle unless the three points lie in a line. There are three choices forming horizontal lines, three choices forming vertical lines, and two choices forming diagonal lines. Thus a total of eight choices out of the 84 possibilities do not give triangles, so there are 76 triangles that can be formed.

18. **Answer B.** Let F be the midpoint of BD and let G be the intersection of CF and DG. Since DE and CG are medians of triangle BCD, it follows that G is the centroid of the triangle, so $CD : DF = 2 : 1$, that is, $GF = \frac{1}{3}CF$. Therefore the area of triangle GFD is one-third of the area of triangle CFD. The latter triangle is one-quarter of the rectangle, so its area is $\frac{1}{4}$, which means that the area of triangle GFD is $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$.

19. **Answer B.** A six-digit number $abcdef$ is divisible by 11 if $f - e + d - c + b - a$ is divisible by 11. (This is because the powers 1, 10, 100, ... alternately have remainders $+1$ and -1 when divided by 11.) Thus $abcdef$ will be divisible by 11 if $(a+c+e) - (b+d+f)$ is equal to 0 or ± 11. Now we know that $(a+c+e)+(b+d+f) = 1+2+3+4+5+7 = 22$, which is even, and

$$[(a+c+e)+(b+d+f)] - [(a+c+e)-(b+d+f)] = 2(b+d+f),$$

so $(a+c+e) - (b+d+f)$ must also be even, which means it cannot be equal to ± 11. Therefore $abcdef$ will be divisible by 11 if $a+c+e = b+d+f$, that is, each side equals 11 since the sum of the two sides is 22. By trial and error, we see that the only solutions of $a+c+e = 11$ are when $\{a,c,e\} = \{1,3,7\}$ or $\{a,c,e\} = \{2,4,5\}$. In each case, there are six ways to arrange a, c, e in order and six ways to arrange b, d, f in order, giving 36 numbers $abcdef$. Since there are two cases to consider, the total is 72 numbers divisible by 11. [Notice that, unlike Question 17, the order of the objects chosen does matter in this question.]

20. **Answer C.** By looking at the angles we see that triangles AED and BFA are similar, as are triangles CFB and DEC. Therefore

$$\frac{BF}{AF} = \frac{AE}{DE} = \frac{3}{5} \text{ and } \frac{BF}{CF} = \frac{CE}{DE} = \frac{7}{5}, \text{ so } \frac{CF}{AF} = \frac{3}{7}.$$

But $CF + AF = CE + AE = 10$, so $CF = 3$ and $AF = 7$. Therefore $BF = \frac{BF}{AF} \times AF = \frac{3}{5} \times 7 = 4.2$.