THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2008

Solutions

1. Answer C. $2008 + 8002 = 10\ 000 + 10 = 10\ 010$.

2. Answer E. $1 + \frac{1}{3+\frac{1}{2}} = 1 + \frac{1}{(\frac{7}{2})} = 1 + \frac{2}{7} = \frac{9}{7}$.

3. Answer D. Area of square $PQRS = 4 \times SOR = 4 \times 16 = 64$. Length of side $PQ = \sqrt{PQRS} = \sqrt{64} = 8$.

4. Answer B. Perimeters are (A) 12, (B) 10, (C) 14, (D) 14, (E) 12. [For any given area, the shape with the smallest perimeter is a circle. Of these five shapes, (B) is the nearest to being a circle.]

5. Answer E. Firstly, $f(2) = 2 \times 2 - 1 = 3$ (one f). Secondly, $f(f(2)) = 2 \times f(2) - 1 = 2 \times 3 - 1 = 5$ (two f's). Thirdly, $f(f(f(2))) = 2 \times f(f(2)) - 1 = 2 \times 5 - 1 = 9$ (three f's).

6. Answer D. The five expressions are (A) Odd + Even, (B) Even + Odd, (C) Odd + Even, (D) Odd + Odd, (E) Even + Odd. [Remember Odd + Even = Even + Odd = Odd, and Odd + Odd = Even + Even = Even.]

7. Answer C. In metres, the circumferences of the two wheels are 2.0 and 2.5, and the distance travelled is 20 000. The numbers of revolutions are $20\ 000 \div 2.0 = 10\ 000$ and $20\ 000 \div 2.5 = 8\ 000$, and the difference between them is $10\ 000 - 8\ 000 = 2\ 000$.

8. Answer A. Firstly, $6 \diamond 4 = \frac{12}{2} = 6$, since the LCM of 6 and 4 is 12, and the HCF of 6 and 4 is 2. Thus $(6 \diamond 4) \diamond 16 = 6 \diamond 16 = \frac{48}{2} = 24$.

9. Answer A. Let x be the number of learners who passed, so $100 - x$ is the number of learners who failed. Since Average mark = (Sum of marks) ÷ (Number of learners), it follows that (Sum of marks) = (Average mark) × (Number of learners). Thus the sum of the marks for the learners who passed is $60x$, the sum for the learners who failed is $30(100 - x)$, and the sum for all learners is $39 \times 100 = 3900$. Thus $60x + 30(100 - x) = 3900$, giving $30x + 3000 = 3900$, so $30x = 900$ and $x = 30$.

10. Answer C. let x be the number of members and y be the number of chairs. Then $\frac{2}{3}x = \frac{3}{4}y$, so $8x = 9y$. Since x and y are integers, this means that x must be divisible by 9, and the smallest positive value is $x = 9$.

11. Answer D. The only numbers divisible by 3 are 3, 9, 15, 21, 27. They are divisible by 3^1, 3^2, 3^1, 3^1, 3^3 respectively, so the product is divisible by $3^{1+2+1+1+3} = 3^8$.

12. Answer E. The slope (or gradient) of the line containing the points $(2, a)$ and $(4, b)$ is $\frac{b-a}{4-2} = -2$, so $b - a = -4$. The slope of the line containing the points $(2, -a)$ and $(4, -b)$ is then $\frac{-b+a}{4-2} = \frac{4}{2} = 2$.

13. Answer E. If a dice is thrown twice, then there are 36 possible outcomes, which we can write as 11, 12, 13…, 64, 65, 66. Of these, there are only four that represent perfect squares, namely 16, 25, 36, 64, so the probability is $4/36 = 1/9$.

14. Answer E. Firstly $A\hat{B}E = 60°$, so $C\hat{B}F = 90° - 60° = 30°$. Next, $B\hat{C}F = \frac{1}{2}B\hat{C}D = 45°$. Finally, $B\hat{F}C = 180° - C\hat{B}F - B\hat{C}F = 180° - 30° - 45° = 105°$.

15. Answer B.
$625^2 \times 32^2 \times 7 = (625 \times 32)^2 \times 7 = (\frac{5}{8} \times 1000 \times 32)^2 \times 7 = (20000)^2 \times 7 = (2 \times 10^4)^2 \times 7$ which has 10 digits (28 followed by eight zeros).

16. Answer B. For each hundreds digit from 1 to 9, the tens digit must be one of its divisors, and then the ones digit is the quotient. We have the following 23 possibilities:

Hundreds:	1	2	3	4	5	6	7	8	9
Tens:	1	1 2	1 3	1 2 4	1 5	1 2 3 6	1 7	1 2 4 8	1 3 9
Ones:	1	2 1	3 1	4 2 1	5 1	6 3 2 1	7 1	8 4 2 1	9 3 1

17. **Answer A.** [A square root can be removed by taking it to one side and then squaring. However, the new equation may have extra solutions that do not belong to the original equation.] Firstly, $\sqrt{x^2 + \sqrt{x^3+1}} = 1 - x$, so $x^2 + \sqrt{x^3+1} = (1-x)^2 = 1 - 2x + x^2$, which simplifies to $\sqrt{x^3+1} = 1 - 2x$. Squaring again gives $x^3 + 1 = (1-2x)^2 = 1 - 4x + 4x^2$, so $x^3 - 4x^2 + 4x = 0$. This can be factorised as $x(x-2)^2 = 0$, which has solutions $x = 0$ and $x = 2$ (repeated). However, by substitution we see that $x = 2$ is not a solution of the original equation, so $x = 0$ is the only solution.

18. **Answer A.** Let O be the centre of the circle (not the square), and let M and N be the midpoints of BC and AD, respectively. If the radius of the circle is r, then $OM = OA = r$, and $AN = 1$. By Pythagoras' theorem $ON = \sqrt{r^2 - 1}$, so $MN = OM + ON = r + \sqrt{r^2-1}$. We also have $MN = AB = 2$, so $r + \sqrt{r^2-1} = 2$, or $\sqrt{r^2-1} = 2 - r$. Squaring gives $r^2 - 1 = (2-r)^2 = 4 - 4r + r^2$, which simplifies to $4r = 5$, so $r = \frac{5}{4}$ (which **is** a solution of the original equation with the square root).

19. **Answer B.** First substitute $x = 0$ to get $f(1) + 2f(0) = 0$ and then substitute $x = 1$ to get $f(0) + 2f(1) = 3$. Multiply the first equation by 2 and subtract the second one to obtain $3f(0) = -3$, which gives $f(0) = -1$.

20. **Answer A.** Let A and B denote the areas of the two shaded regions, and let P be the area of one of the unshaded regions between them. The semicircle has area $S = \frac{1}{2}\pi 1^2 = \frac{1}{2}\pi$, each quarter circle has area $Q = \frac{1}{4}\pi 2^2 = \pi$, and the square has area $2^2 = 4$. Then from the diagram we see that $S + A + P = Q$ and $B + P + Q = 4$. Subtract the second equation from the first to obtain $A - B = 2Q - S - 4 = \frac{3}{2}\pi - 4$.