THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION
Sponsored by HARMONY GOLD MINING

FIRST ROUND 2007

SENIOR SECTION: GRADES 10, 11 AND 12

20 MARCH 2007

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. Answer D
2. Answer B
3. Answer E
4. Answer D
5. Answer E
6. Answer D
7. Answer E
8. Answer B
9. Answer E
10. Answer A
11. Answer D
12. Answer C
13. Answer D
14. Answer B
15. Answer D
16. Answer B
17. Answer B
18. Answer E
19. Answer D
20. Answer E

SOLUTIONS

1. **Answer D.** R16 for 80 grams means R2 for 10 grams, therefore R200 for 1000 grams, which is one kilogram.

2. **Answer B.** There are $80 \times 3 = 240$ sandwiches. The first people take five sandwiches each, so the sandwiches are finished after $240 \div 5 = 48$ people have helped themselves. That leaves $80 - 48 = 32$ people who do not get any sandwiches.

3. **Answer E.** If the attendance doubles every school day, then each day's attendance is half of the next day's attendance (until the class is full). Therefore the class was half full one day earlier, on 17 January.

4. **Answer D.** For each 100 mℓ, the fat content is reduced from 3.4 g to 0.5 g, so 2.9 g is removed. The proportion of fat that is removed is $2.9/3.4 \approx 0.85 = 85\%$.

5. **Answer E.** First, $a_2 = 2a_1 + a_0 = 2 \times 0 + 2 = 2$, then $a_3 = 2a_2 + a_1 = 2 \times 2 + 0 = 4$, then $a_4 = 2a_3 + a_2 = 2 \times 4 + 2 = 10$, and finally $a_5 = 2a_4 + a_3 = 2 \times 10 + 4 = 24$.

6. **Answer D.** The sum of all 25 numbers in the magic square is $1 + 2 + 3 + \cdots + 24 + 25 = \frac{1}{2} \times 25 \times 26 = 325$. The 25 numbers form five rows (or five columns), all of which have the same sum, so the sum of the numbers in one row (or column) is $325 \div 5 = 65$.

7. **Answer E.** The radius $r = OD = \sqrt{OE^2 + ED^2} = \sqrt{OE^2 + (\frac{1}{2}CD)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. (Note that $ED = \frac{1}{2}CD$ since E is the midpoint of CD.) The circumference $= 2\pi r = 4\pi\sqrt{5}$.

8. **Answer B.** Square both sides to get $x^4 + 16 = (x^2 + 4)^2 = x^4 + 8x^2 + 16$, which simplifies to $8x^2 = 0$, for which the only solution is $x = 0$.

9. **Answer E.** If 20 numbers have an average of 20, then their sum is $20 \times 20 = 400$. If 9 of those numbers have an average of 9, then their sum is $9 \times 9 = 81$. Thus the sum of the remaining 11 numbers is $400 - 81 = 319$, so their average is $319 \div 11 = 29$.

10. **Answer A.** If you subtract each term from the next term, then you obtain the sequence $6;\ -2; 6;\ -2; 6;\ -2; x-15;\ 19-x;\ 6; \ldots$. It therefore appears that $x-15 = 6$ and $19-x = -2$, both of which give the same value $x = 21$.

11. **Answer D.** This does not need to be solved exactly — an approximate solution is sufficient. Suppose the raised bar forms two right-angled triangles. If the distance between the towns is $2d$ and the extra length is $2x$, then the base of one triangle is d and the hypotenuse is $d + x$. The height h is given by $h^2 = (d + x)^2 - d^2 = 2dx + x^2 \approx 2dx$, since x is much smaller than d. Therefore $h \approx \sqrt{2dx}$. Working in metres, we have $d = 10\ 000$ and $x = 0.5$, so $h \approx \sqrt{2 \times 10\ 000 \times 0.5} = \sqrt{10;000} = 100$.

12. **Answer C.** Each fraction is of the form $\frac{n}{n+1} = 1 - \frac{1}{n+1}$. The smallest fraction therefore corresponds to the biggest value of $\frac{1}{n+1}$, which corresponds to the smallest denominator, which is 20042005.

13. **Answer D.** Let O be the centre of the circle, let P be the centre of the small circle, and let Q be the point where the two circles touch. If the radius of the small circle is r, then $0P = r\sqrt{2}$ (diagonal of a square), and $PQ = r$. Since $OP + PQ = OQ = 8$, it follows that $r(\sqrt{2} + 1) = 8$, so $r = \dfrac{8}{\sqrt{2} + 1}$ (which can be simplified to $8(\sqrt{2} - 1)$).

14. **Answer B.** Join the centres of the three extreme circles to form an equilateral triangle whose sides are of length 20, so by Pythagoras its height is $10\sqrt{3}$. The figure extends one radius above and below this triangle, so the height of the figure is $10 + 10\sqrt{3}$.

15. **Answer D.** It is easy to guess the mother's age, so you can find the answer by trial and error. Here is a more focused method. Suppose the mother's age is m and the daughter's age is d. In forming the four-digit number, the mother's age is moved two places to the left, which is the same as multiplying it by 100, so the number is $100m + d$. We then subtract $m - d$ to obtain 4202, so we have the equation $(100m + d) - (m - d) = 4202$. This simplifies to $99m + 2d = 4202$ or $99m = 4202 - 2d$. It follows that $4202 - 2d$ must be divisible by 99, and the only possibility is $4202 - 2d = 42 \times 99 = 4158$, so $d = (4202 - 4158) \div 2 = 22$.

16. **Answer B.** Let the unknown square on the top and right have sides of length x, the square on the left have sides of length y, and the square on the bottom have sides of length z. Then from the top and bottom we have $14 + x = 9 + z + 15$, so $x - z = 10$. Similarly, from the left and right we have $14 + y + 9 = x + 15$, so $x - y = 8$. Then using the 4×4 square in the middle we see that $x - 4 = 14$, so $x = 18$. We then have $y = 10$ and $z = 8$, so the rectangle is 33 high by 32 wide, and its area is $33 \times 32 = 1056$. [You can check the answer by adding up the areas of all the squares if you really want to.]

17. **Answer B.** Triangle ABC is right-angled (angle in a semicircle), so by Pythagoras $AB = \sqrt{41^2 - 40^2} = \sqrt{81} = 9$. The shaded region is obtained by removing the large semicircle from the region formed by the triangle and the two smaller semicircles, so (using the formula Area of semicircle $= \frac{\pi}{8}d^2$, we have

$$\text{Area of shaded region} = \text{Area of triangle} + \frac{\pi}{8}(9^2 + 40^2 - 41^2) = \text{Area of triangle},$$

using Pythagoras' theorem again. Finally, the area of the triangle is $\frac{1}{2} \times 9 \times 40 = 180$. [The two parts of the shaded region are called *lunes*, because they look like crescent moons. The fact that the combined area of the two lunes is equal to the area of the triangle is a theorem due to the ancient Greek mathematician Hippocrates.]

18. **Answer E.** Since the three numbers each leave the same remainder after division by D, it follows that the difference between any two of them must be divisible by D. Thus D is a common factor of $227 - 128 = 99$ and of $128 - 73 = 55$. The only possibility is $D = 11$ (since we are given that $D > 1$), and $73 = 6 \times 11 + 7$ so $R = 7$. Thus $D - R = 11 - 7 = 4$. [Numbers that leave the same remainder after division by D are said to be congruent modulo D, and we write $227 \equiv 128 \bmod 11$.]

19. **Answer D.** Suppose the large cube has n small cubes along each side, so it contains a total of n^3 small cubes. Since one layer of small cubes is visible on each face, it follows that there are $(n - 2)^3$ invisible small cubes. We therefore need the smallest value of n such that $(n - 2)^3 > \frac{1}{2}n^3$, or $n^3 < 2(n - 2)^3$. One way is to draw up a table:

n	1	2	3	4	5	6	7	8	9	10	11
n^3	1	8	27	64	125	216	343	512	729	1000	1331

The first time that a number in the second row is less than twice the number two places before it is when $n = 10$, because $1000 < 2 \times 512$ but $729 > 2 \times 343$. Thus there are 1000 small cubes in the large cube.

20. **Answer E.** Triangle CDE is similar to triangle DGE, so $DG/GE = CD/DE = 2$, since we are given that E is the midpoint of DA. The area of triangle DGE is $\frac{1}{2} \times DG \times GE = \frac{1}{2} \times 2GE \times GE = GE^2$. This area is equal to 1, so $GE = 1$, which gives $DG = 2$, and by Pythagoras $DE = \sqrt{5}$. Therefore each side of the square has length $2\sqrt{5}$, and the area of the square is $(2\sqrt{5})^2 = 20$.