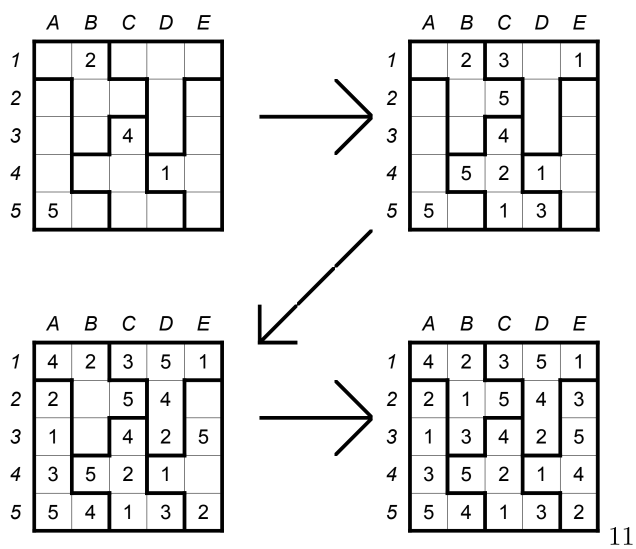
The South African Mathematical Olympiad

Junior Third Round

Solutions

1. Let x be the number that is the average of the other four. The sum of the other four numbers will then be four times their average, i.e. equal to $4x$. Then the sum of all 5 numbers will be equal to $4x + x = 5x$, so x is also the average of all five numbers. The average of the given numbers is $\frac{26+30+37+28+29}{5} = \frac{150}{5} = 30$, which is the number we want.

2. Label the columns A to E and the rows 1 to 5 as shown. We have $D4 = 1$, which means that $C5 = 1$, since the 1 can go nowhere else in that puzzle piece, and it follows that $E1 = 1$. In that same puzzle piece, the only place where 2 can go is $D2$ or $D3$. In particular, it means that $D5$ cannot equal 2. Also, $B4$ cannot equal 2, so $C4 = 2$. Now $B4 = 5$ and $D5 = 3$ follows easily, from which we get that $C2 = 5$ and in column C, $C1 = 3$.



Looking at column B, we see that 1 and 3 can only go into squares $B2$ or $B3$. This means that $B5 = A1 = 4$, from which it follows that $E5 = 2$. Looking at column E, $C2 = B4 = 5$ forces $E3 = 5$, which then forces $D1 = 5$. $D2 = 4$ and $D3 = 2$ now follows easily. In column A, $C4 = D3 = 2$ now forces $A2 = 2$, which forces $A3 = 1$ and $A4 = 3$. The rest of the square is now easily completed.

3. The hour hand of the clock moves through an angle of $360°$ in 12 hours, which means that it moves $\frac{360°}{12} = 30°$ per hour. At 19:30, the hour hand has been moving for 7 and a half hours since noon, so it has moved through an angle of $30° \times 7.5 = 225°$ clockwise, measured from the line joining the centre of the clock and the 12. The minute hand is exactly on

the 6, so the angle between it and the line joining the 12 to the centre of the clock is equal to $180°$. The angle between the two hands is thus equal to $225° - 180° = 45°$.

Alternative solution

Half past 7 is exactly between six o' clock and nine o' clock. The angle between the digit 6 and the digit 9 on the clock (with respect to the centre of the clock) is equal to 90 degrees, so the angle between the hour hand and the digit 6 on the clock is $\frac{1}{2} \times 90° = 45°$. But the at 19:30, the minute hand is exactly on the 6, so the angle between the two hands is equal to $45°$.

4. Inspecting the diagram, and remembering that all the squares are 2×2, we see that G is on top of D, which is on top of E, which is on top of both H and I. Because I occupies the corner, H must lie on top of I. Similarly, square C occupies the corner, so E and F both lie on top of C, and also E lies on top of F. Square A also occupies a corner, so it lies below both B and D, which then means that B must lie below both D and C as well. Putting all this information together, and noting that since F lies on top of C, F must lie under I, we get that the squares were put down in the order $A, B, C, F, I, H, E, D, G$.

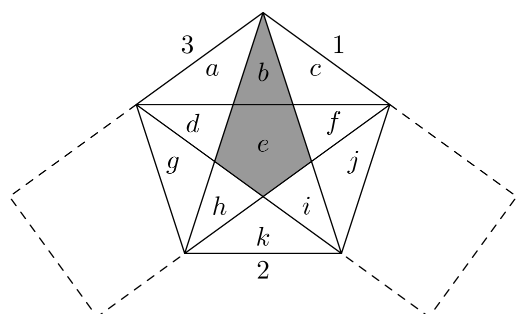
5. (a) Let the first integer of the 28 be equal to n. We wish to find n such that
$$n + (n + 1) + \cdots + (n + 27) = 294.$$
The left hand side simplifies to
$$28n + (1 + 2 + \cdots + 27) = 28n + \frac{27 \cdot 28}{2} = 28n + 378.$$
The equation becomes $28n + 378 = 294$, or $28n = -84$ which gives $n = -3$. So
$$-3 + (-2) + \cdots + 24 = 294.$$
   (b) To find three such consecutive integers, we need to solve the equation $n+(n+1)+(n+2) = 50$, which simplifies to $3n+3 = 50$, or $3n = 47$. However, 47 isn't a multiple of 3, so there is no solution.

6. (a) False. π is an irrational number, while $\frac{22}{7}$ is a rational number.
   (b) Let the radius of the circle be equal to r. Then the side length of the square is equal to $2r$. The perimeter of the shaded region consists of a quarter circle, together with two halves of the sides of the square. We know this perimeter equals 25 cm, so $25 = \frac{1}{4}(2\pi r) + 2 \times \frac{1}{2} \times (2r) = \frac{\pi r}{2} + 2r = r(\frac{\pi}{2} + 2)$. Using the estimate $\pi \approx \frac{22}{7}$, this becomes $25 = r(\frac{11}{7} + 2) = r(\frac{25}{7})$ which gives $r = 7$. The area of the circle is thus equal to $\pi r^2 \approx \frac{22}{7} \times 7^2 = 154$ cm^2.

7. Let $\angle ADB = \angle ABD = x$ and $\angle ACB = y$. Then, in triangle BCD, the external angle $\angle ADB = \angle DCB + \angle CBD$, the sum of the opposite internal angles. This equation becomes $x = y + \angle CBD$, so $\angle CBD = x - y$. The given equation then becomes $30° = \angle ABC - \angle ACB = (x + x - y) - y = 2(x - y)$, so $\angle CBD = x - y = 15°$.

8. If K leaves a remainder of 5 when divided by 6, it means that K is one less than a multiple of 6. So if K leaves remainders of 5, 4, 3, 2, 1 when divided by 6, 5, 4, 3, 2, respectively, it means that K is one less than a multiple of 6, 5, 4, 3 and 2. The smallest such number is one less than the least common multiple of 6, 5, 4, 3 and 2, which equals $2^2 \times 3 \times 5 - 1 = 59$.

9. (a) A pentagon.



11

(b) Label the regions and edges of the pentagon as shown. We can reproduce the pentagon in the knot by folding the strip of paper downwards at edge 1, folding it upwards at edge 2 and folding it downwards again at edge 3.

Before the first fold, the regions that are covered by the strip of paper are a, b, c, d, e, f, g and h. After folding the strip of paper downwards at edge 1, the regions that are covered are b, c, e, f, h, i, j and k. After folding the strip of paper upwards at edge 2, the regions that are covered are a, b, d, e, g, h, i and k. Finally, folding the strip of paper downwards at edge 3 covers the regions a, b, c, d, e, f, i and j.

The only regions covered by all four layers are the regions b and e, which together form a kite.

10. (a) The sequence of last terms on the right hand side is $3, 8, 15, 24, \ldots$, which are all one less than a perfect square. The formula for the last number on the right hand side of row n is $(n+1)^2 - 1$.

Better solution

In the first row there are 3 numbers. In the second row there are 5 numbers, and so on, so in the n^{th} row there are $2n + 1$ numbers. So the last term on the right hand side of the n^{th} row is equal to

$$3 + 5 + \cdots + (2n + 1) = [1 + 3 + 5 + \cdots + (2n + 1)] - 1 = (n + 1)^2 - 1$$

by the well-known formula $1 + 3 + \cdots + 2n - 1 = n^2$.

(b) From the given pattern it is evident that there are n numbers on the right hand side of row n. Since the last term on the right hand side is equal to $(n+1)^2 - 1$, the last term on the left hand side is equal to $(n+1)^2 - 1 - n = n^2 + n$.

(c) The last term on the left hand side of row n is n^2+n, and the last term on the right hand side is equal to $(n+1)^2-1 = n^2+2n = (n^2+n)+n$. The sum on the right hand side of row n is thus equal to

$$\begin{aligned}
& (n^2+n+1) + (n^2+n+2) + \cdots + (n^2+n+n) \\
= \; & n(n^2+n) + (1+2+\cdots+n) \\
= \; & n^2(n+1) + \frac{n(n+1)}{2} \\
= \; & n(n+1)(n+\tfrac{1}{2}) \\
= \; & \frac{n(n+1)(2n+1)}{2}.
\end{aligned}$$

11. (a) 49 997 ends in an odd number, so it passes through Loop 2 with output $4+9+9+9+7 = 38$. 38 ends on an even, non-zero number, so it passes through Loop 1 with output $\frac{38}{2} = 19$. 19 ends on an odd number, so it passes through Loop 2 with output $1+9 = 10$. 10 ends on 0, so the process stops with final output 10 after passing through Loop 1 once and through Loop 2 twice.

(b) Any even number not ending on a 0 will pass through Loop 1, dividing it by 2. If a number didn't end on a 0, then dividing it by 2 will not make it end on a 0, so the process will continue. The only way in which the process will not pass through loop 2 is if the resulting numbers never end on an odd digit. However, this is impossible, since any even number only has a finite number of factors of 2. So any even number not ending on a 0 will pass through both loops at least once.

(c) Any odd number will pass through Loop 2, with the result equal to the sum of the digits of the odd number. Note immediately that if n is one of 1, 3, 5, 7 or 9, the number will just keep passing through Loop 2. If $n > 10$, then the only way that the process will not pass through Loop 1 is if the digit sum is odd or ends on a zero. However, the biggest possible digit sum of a number less than 150 is equal to $9 + 9 = 18$, so if the digit sum ends on a 0, it must in fact equal 10. The odd numbers less than 150 with digit sum equal to 10 are

$$19, 37, 55, 73, 91, 109, 127, 145.$$

Since $1, 3, 5, 7, 9$ never pass through Loop 1, any odd number with digit sum equals one of them will not pass through Loop 1 either.

The odd numbers less than 150 with digit sum equal to 1, 3, 5, 7 or 9 are

$$1, 3, 5, 7, 9, 21, 23, 25, 27, 41, 43, 45, 61$$
$$63, 81, 111, 113, 115, 117, 131, 133, 135.$$

The positive odd numbers less than 150 that go through both loops at least once are all the odd numbers less than 150 not in the above two lists.

12. Suppose that Bongani wrote n Maths tests before his last test. Since his average is 33% in those n tests, the sum of his scores in those tests equal $33n$ percent. He then scored 40% in his next test to increase his average to 34%. The sum of all his scores is then equal to $33n + 40$ percent, giving an average of $\frac{33n+40}{n+1} = 34$ percent. Solving this equation for n gives $n = 6$.

    Thus, if Bongani wants an average of 35% after $6 + 2 = 8$ tests, the sum of his scores should equal $8 \times 35\% = 280\%$. After 7 tests the sum of his scores equal $7 \times 34\% = 238\%$, so he needs to score $280\% - 238\% = 42\%$ in his eighth test.

13. There are all together 6 bags containing other bags, including the outermost bag. So 5 of the 11 bags Fred put inside the outermost bag must also contain other bags. Hence there are a total of $1 + 11 + 5 \times 11 = 67$ bags. Since 6 bags contain other bags, $67 - 6 = 61$ remain empty.

14. Note that 20 on the far-away planet corresponds to our $26 = 2 \times 13$, 30 on the far-away planet corresponds to our $39 = 3 \times 13$, and so on. This means that $100 = 10 \times 10$ on the faraway planet corresponds to our $13 \times 13 = 169$.

    The number $1x$ corresponds to our $13 + 4 = 17$. The square of $1x$ in our system is thus equal to $17^2 = 289$, which we can write as $289 = 169 + 117 + 3 = 13^2 + 9 \times 13 + 3$. This corresponds to the far-away planet's $10^2 + 7 \times 10 + 3 = 173$.

15. Suppose Mr Mahlanyana had g girls as grandchildren, and each boy got an amount of Rx. Then there are $31 - g$ boys and so the total legacy of the grandchildren equals R470 $= x(31 - g) + (7 + x)g = 31x + 7g$, so $x = \frac{470-7g}{31}$.

    Now, if Mrs Zweni has g_1 girls and b_1 boys, then the legacy going to Mrs Zweni's children equals

    $$\text{R}74 = b_1x + g_1(7 + x) = (b_1 + g_1)x + 7g_1 = (b_1 + g_1)\frac{470 - 7g}{31} + 7g_1.$$

    We see from the equation that since 74 is an integer and $7g_1$ is an integer, $(b_1 + g_1)\frac{470-7g}{31}$ must also be an integer. But since 31 is a prime number, and $b_1 + g_1 < 31$ (Mrs Zweni cannot have more children than Mr Mahlanyana has grandchildren!), 31 cannot cancel with $b_1 + g_1$, so $470 - 7g$

is a multiple of 31. We see that 470 leaves a remainder of 5 when divided by 31, which means that $7g$ must also leave a remainder of 5 when divided by 31. The smallest multiple of 7 leaving a remainder of 5 when divided by 31 is $98 = 7 \times 14$. So $g = 14$, and $x = \frac{470-7\times 14}{31} = 12$.

So, each boy gets R12 and each girl gets R19. Thus $12b_1 + 19g_1 = 74$. The only possible values for g_1 are 0, 1, 2 or 3. Since 12 and 74 are both even but 19 is odd, we see that g_1 must also be even. If $g_1 = 0$, the equation becomes $12b_1 = 74$ which is impossible since 74 is not a multiple of 12. If $g_1 = 2$, then $12b_1 = 74 - 19 \times 2 = 36$ giving $b_1 = 3$. Thus Mrs Zweni has 2 daughters.