

**Harmony South African Mathematics Olympiad
Solutions: Third round Junior, 2008
Junior Section (Grades 8 & 9)**

Question 1

Let 'x' be the number of packs containing 5 sweets and 'y' the number of packs containing 7 sweets.

$$\therefore 5x + 7y = 106 \quad \textcircled{1}$$

$$\therefore \text{Cost of } x \text{ packets} = 6x$$

$$\text{Cost of } y \text{ packets} = 7y$$

But $6x + 7y$ to be minimum

$$\text{Let } 6x + 7y = p \quad \textcircled{2}$$

When 'p' is the minimum cost.

$$\text{From } \textcircled{1} \text{ and } \textcircled{2} \quad x = p - 106$$

$\therefore p$ must be more than 106.

$$7y = p - 6(p - 106) \quad \therefore 7y = -5p + 636$$

$$\text{When } p = 107, \quad y = \frac{101}{7} \quad x = 1$$

$$\text{When } p = 108 \quad y = \frac{96}{7} \quad x = 2$$

$$\text{When } p = 109 \quad y = 13 \quad x = 3 \quad \text{Which are whole numbers}$$

\therefore Number of packs of 7 sweets = 13

\therefore Number of packs of 5 sweets = 3

Minimum cost = 109

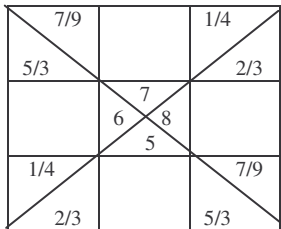
OR:

Let x be the number of packs containing 5 sweets, and y the number of packs containing 6 sweets. Then $5x + 7y = 106$. The total cost is $6x + 7y = 106 + x$. For the minimum cost, clearly x must be a minimum. We find x by inspection, but we can save some time considering divisibility by 7 (i.e. arithmetic modulo 7). Write the first equation as $5x - 1 = 105 - 7y = 7(15 - y)$, from which it follows that $5x - 1$ must be divisible by 7. We see quickly that $x = 3$ is the smallest solution, so the minimum cost is $106 + 3 = 109$.

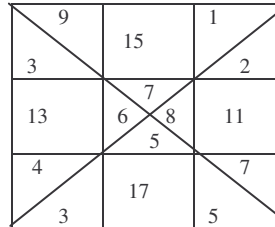
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Question 2

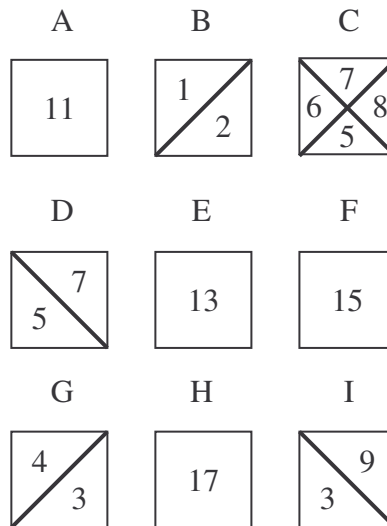
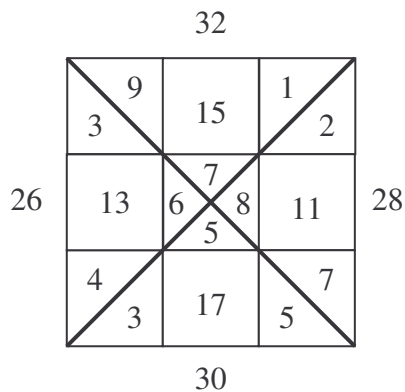
Clearly there is only one possibility for the centre square, and two possibilities for each corner square so we have



Now by trial and error we can place the remaining blocks to ensure the correct sum for each triangular region, and we obtain



OR



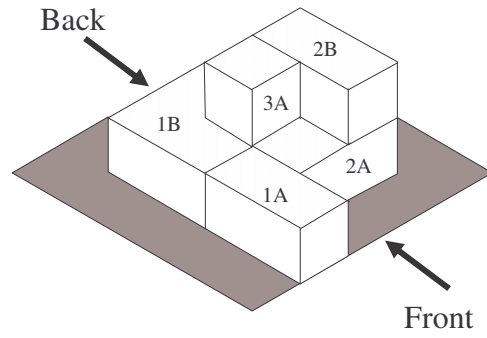
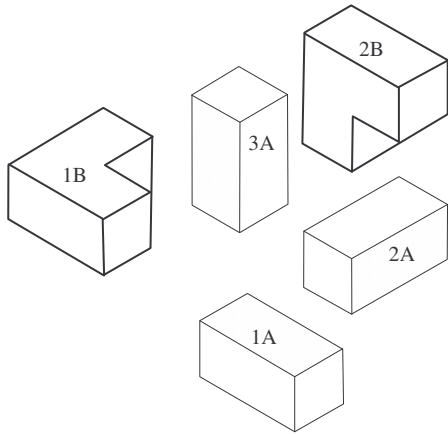
- Clearly, block C must be placed in the middle box.
- The top left box can only be occupied by D or I and the bottom left box can only be occupied by block B or block G.
- Place block I in the top left box and G in the bottom left box. Adding the numbers in the left big triangle, we have $3 + 6 + 4 = 13$. In order to have a sum of 26, we must place block E in the middle left box.
- Place block B in the top right box.
[If we had place block G in that box, the sum of the three numbers would be $9 + 7 + 4 = 20$. In order to get 32 in that top big triangle we would need to fill the middle top box by a block with a 12. We do not have such a block.]
In order to get 32 in the top “big” triangle, we need a block with $32 - (9 + 1 + 7) = 15$. Place block F in the top middle box.
- It is now obvious that block A must be placed in the middle right block, and block D in the bottom right box. Check that the numbers in the left “big triangle add up to 28: $2 + 8 + 11 + 7 = 28$

[4]

Question 3

Block A = 3

Block B = 2



[6]

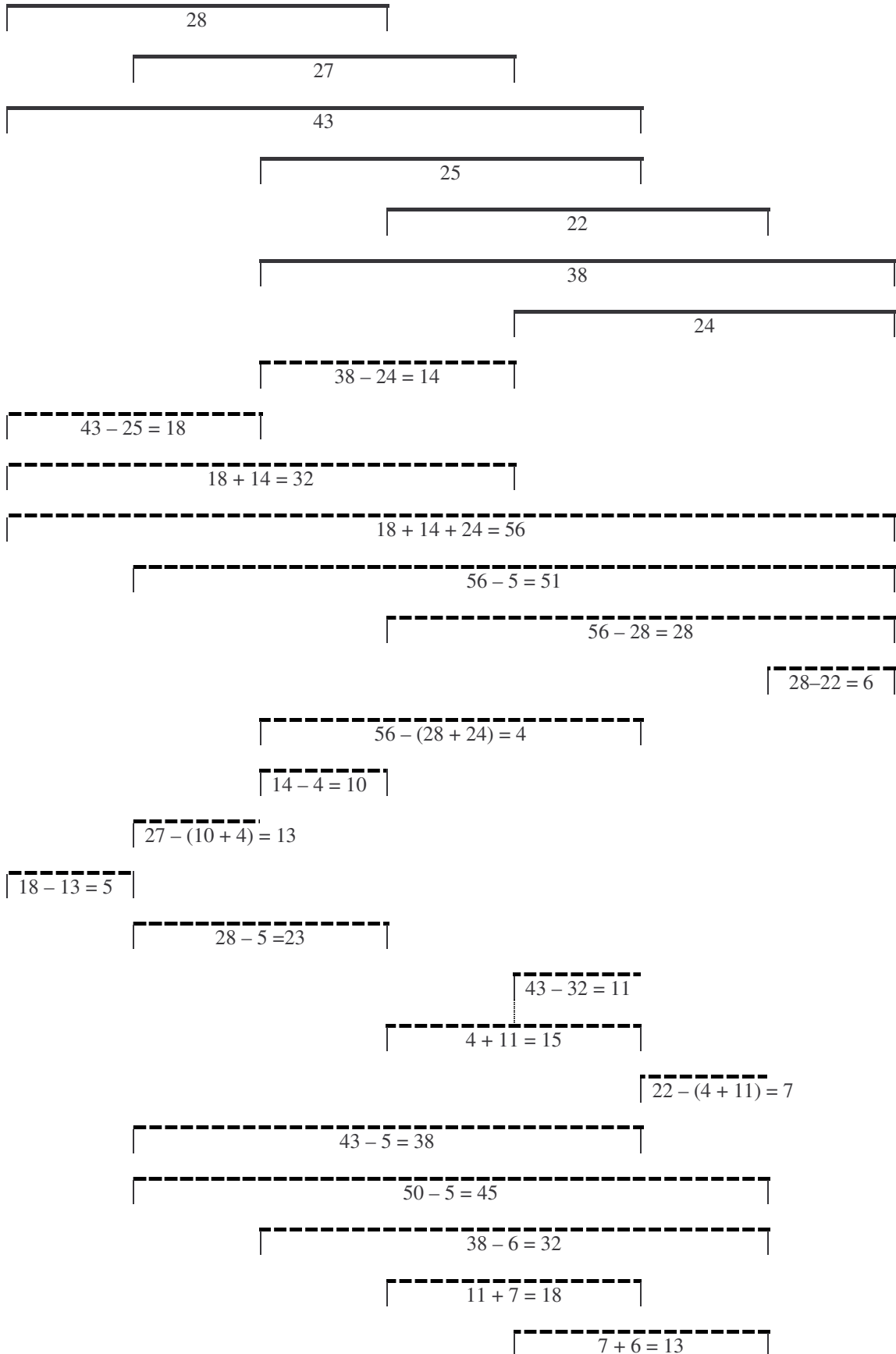
Question 4

				0		
				2		
			*	*		2
1	*	*	*	*	*	*
		4	*	*	4	
		1	2	2	1	

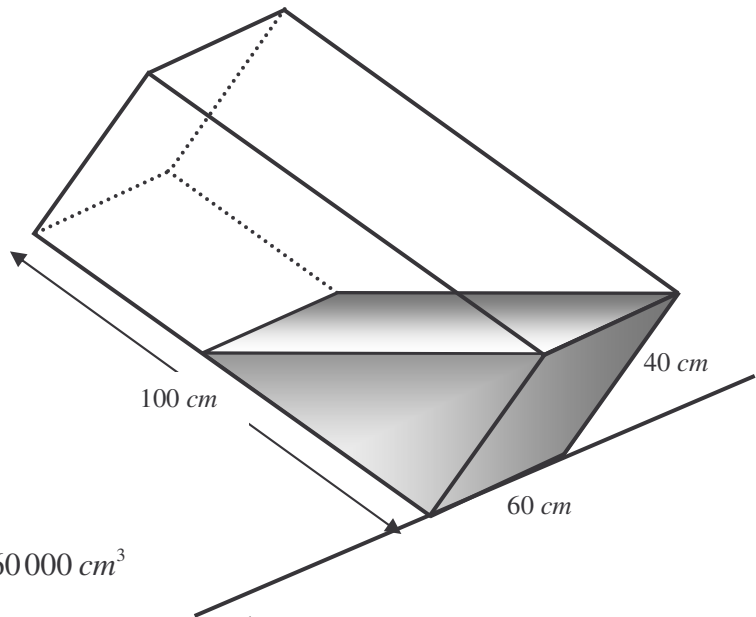
[6]

Question 5

A B C D E F G H



Question 7



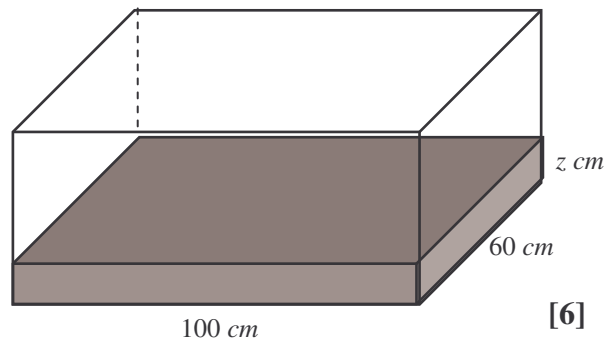
Volume of water $\frac{1}{2} \times 50 \times 40 \times 60 = 60\,000 \text{ cm}^3$

Or $\frac{1}{4} \times 100 \times 40 \times 60 = 60\,000 \text{ cm}^3$ [$\frac{1}{4}$ of total]

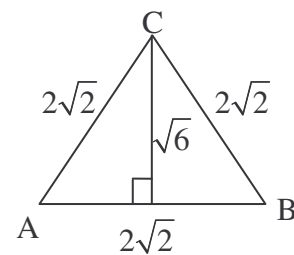
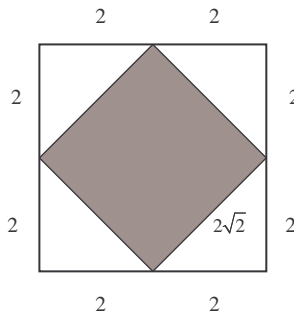
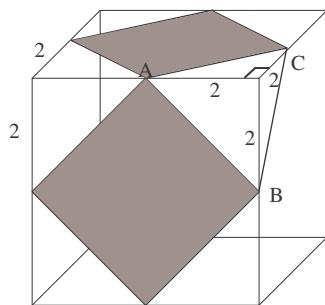
Let z be the height reached by the water.

Then $60\,000 = 100 \times 60 \times z$

$$z = \frac{60\,000}{6\,000} = 10 \text{ cm}$$



Question 8



The length of the line joining the midpoints of two adjacent sides of the cube is $2\sqrt{2}$, so the cuboctahedron has six square faces (left over from the six faces of the cube) and eight triangular faces (where the eight vertices of the cube have been cut off).

Area of the shaded region $= 2 \times \frac{1}{2} \times 4 \times 2 = 8 \text{ cm}^2$

Since there are six sides in a cube, the total surface area of the six “diamond regions” is $6 \times 8 \text{ cm}^2 = 48 \text{ cm}^2$. The area of triangle ABC $= \frac{1}{2} \times 2\sqrt{2} \times \sqrt{6} = \sqrt{12} = 2\sqrt{3} \text{ cm}^2$. There are eight such triangular regions resulting from cutting from each of the eight corners of the cube. The total area of the triangular regions is $8 \times 2\sqrt{3} \text{ cm}^2 = 16\sqrt{3} \text{ cm}^2$.

Therefore the total surface area of the cuboctohedron is $48 \text{ cm}^2 + 16\sqrt{3} \text{ cm}^2$. [6]

Question 9

Special case: 3; 4; 5 gets 0.

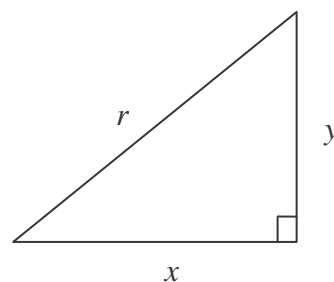
$$x^2 + y^2 = r^2,$$

$$P = x + y + r$$

Perimeter

$$A = \frac{1}{2}xy$$

Area



$$\begin{aligned} xP &= x^2 + xy + xr \\ yP &= xy + y^2 + yr \\ (x + y)P &= x^2 + y^2 + 2xy + (x + y)r \\ (P - r)P &= r^2 + 4A + (P - r)r \\ P^2 - rP &= r^2 + 4A + Pr - r^2 \\ P^2 - rP &= 4A + Pr \\ P^2 - 4A &= 2Pr \\ r &= \frac{P^2 - 4A}{2P} \end{aligned}$$

Or :

$$\begin{aligned} r &= P - (x + y) \\ \text{but } xy &= 2A \\ \text{and } x^2 + y^2 &= r^2 \\ \Rightarrow (x + y)^2 &= r^2 + 4A \\ \therefore (P - r)^2 &= r^2 + 4A \\ \therefore r &= \frac{P^2 - 4A}{2P} \end{aligned}$$

Or :

$$\begin{aligned} P^2 &= x^2 + y^2 + r^2 + 2xy + 2rx + 2ry \\ &= 2r^2 + 4A + 2rx + 2ry \\ \therefore 2r(r + x + y) &= P^2 - 4A \\ \therefore 2r(P) &= P^2 - 4A \\ \therefore r &= \frac{P^2 - 4A}{2P} \end{aligned}$$

[8]

Question 10

Digits that are mandatory will be 1; 4; 9.

An even number with four factors must be of the form $2p$, where p is a prime greater than 2. (Three of its factors are 1, 2 and itself, so there is only one other factor.) Now try out the possibilities:

Even numbers with 4 factors are;
 14; 34; 74; 94 (all reversible into primes except 94)
 Sum of digits is 2-digit prime from 74 [7 + 4 = 11];
 therefore answer is 74 as it fits all the properties.

p	$2p$	Reverse digits	Sum of digits	Square digit
5	10	01X	1X	Yes
7	14	41Yes	5X	Yes
11	22	22X	4X	X
13	26	62X	8X	X
17	34	43Yes	7X	Yes
19	38	83Yes	11Yes	X
23	46	64X	10X	Yes
29	58	85X	13Yes	X
31	62	26X	8X	X
37	74	47Yes	11Yes	Yes
47	94	49X	13Yes	Yes

[8]

Question 11

If 2 was one of the incorrect divisors then 6 would be as well so we can rule out 2 (since 2 and 6 are not consecutive). Similarly we can rule out 3, 4, 5 and 6 as being one of the incorrect divisors since 6, 8, 10, and 12 (respectively) would be one of the incorrect divisors. We can also rule out $10 = 5 \times 2$ and $12 = 3 \times 4$, since if 10 or 12 are not divisors then one of 5 or 2 or one of 4 or 3 can't be either. This means that 11 and 13 can't be incorrect divisors either. So the only 2 possibilities for the consecutive non-divisors are {7,8} or {8,9}. If they were {8,9} then 7 would be a factor and the smallest number with all above factors is $5 \times 7 \times 11 \times 12 \times 13 = 60060$ which is larger than 50 000. So the 2 consecutive non-divisors are {7,8} and the number is $3 \times 5 \times 11 \times 12 \times 13 = 25740$

[8]

Question 12

$$\begin{aligned}t_1 &= 1; & t_2 &= 2 \\t_3 &= \left(\frac{3-3}{3-1}\right)t_1 = 0; & t_4 &= \frac{1}{3}t_2 = \frac{1}{3}(2) = \frac{2}{3}, \\t_5 &= 0; & t_6 &= \frac{3}{5}\left(\frac{1}{3}\right)t_2 = \frac{1}{5}(2) = \frac{2}{5}\end{aligned}$$

Odd Members

$$t_{2n+1} = 0 \text{ for } n > 0$$

Even Members

$$\begin{aligned}t_{2n} &= t_2 \left(\frac{1}{\cancel{3}}\right) \left(\frac{\cancel{3}}{\cancel{3}}\right) \left(\frac{\cancel{3}}{\cancel{3}}\right) \cdots \left(\frac{\cancel{2n-3}}{2n-1}\right) \text{ for } n > 0 \\&= \frac{t_2}{2n-1} = \frac{2}{2n-1}\end{aligned}$$

(a) From the above $t_{2007} = 0$

(b) $t_{2008} = \frac{2}{2008-1} = \frac{2}{2007}$

Question 13

(a) $10a + b = ab + a + b$
 $9a = ab$
 $a(b-9) = 0$
 $a = 0$ or $b = 9$

Since the first digit a cannot be 0, it follows that b must be 9. Thus any 2-digit number ending in 9 has this property:

19; 29; 39; 49; 59; 69; 79; 89; 99

(b) Since $a \neq 0$, we can divide through to get $99 + 9\frac{b}{a} = bc$. There are no solutions, since the left-hand side is ≥ 99 and the right-hand side is ≤ 81 .

OR

- (b) For a 3-digit number to have the required property,
 $100a + 10b + c = abc + a + b + c$

$$\therefore 99a + 9b = abc$$

$$\therefore \frac{99}{b} + \frac{9}{a} = c$$

but $b \leq 9$

$$\Rightarrow \frac{99}{b} \geq 11$$

$$\therefore c \geq 11$$

not possible as c must be a single digit.

[8]

Question 14

(a) $\sqrt{4 \times 3 \times 2 \times 1 + 1} = 5$ ($= 4 \times 1 + 1$)

(b) $\sqrt{51 \times 50 \times 49 \times 48 + 1} = 2449$ ($= 51 \times 48 + 1$)

- (c) If the 4 consecutive integers are $n-1, n, n+1, n+2$ we observe that the square root of $(n-1)(n)(n+1)(n+2)+1$ has to lie between n^2 and $(n+1)^2$. The 2 combinations that give this number are $(n-1)(n+2)+1$ or $(n)(n+1)+1$ (where we add 1 to make the number odd). From the above we observe that the correct combination is $(n-1)(n+2)+1 = n^2 + n - 1$ so that $\sqrt{(n-1)(n)(n+1)(n+2)+1} = n^2 + n - 1$ and the product is a perfect square.

OR: Multiply $(n+2)(n+1)(n)(n-1)+1$ and then used trial and error to find the square root, which had to be of the form $n^2 + bn \pm 1$

Or: A more formal proof: $(n-1)(n)(n+1)(n+2)+1 = \left(m - \frac{3}{2}\right)\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right)\left(m + \frac{3}{2}\right) + 1$

Let $m = n + \frac{1}{2}$ then

$$= \left(m^2 - \frac{9}{4}\right)\left(m^2 - \frac{1}{4}\right) + 1$$

$$= m^4 - \frac{10}{4}m^2 + \frac{25}{16}$$

$$= \left(m^2 - \frac{5}{4}\right)^2$$

$$= \left(\left(n + \frac{1}{2}\right)^2 - \frac{5}{4}\right)^2$$

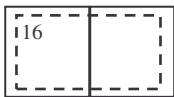
$$= (n^2 + n - 1)^2$$

$$\therefore \sqrt{(n-1)(n)(n+1)(n+2)+1} = n^2 + n - 1$$

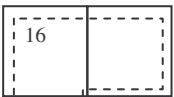
[8]

Question 15

Start with this block (top left):



It can only be 7 and 9



Therefore this block can only be 2, 6 and 8



And this block 3 and 4

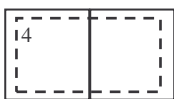


Can only be 1 and 3 with 3 in block E for 3 is already used.



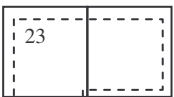
16		7	4		23		8	
16		A	9		B	13		C
	4	19	15		5		18	16
5	E			6				
	20	5		12		17		
15		15		6	12		19	6
		7						
8	6	16		23		13		10
					11			

If we move to the top middle block:



Can only be 1 and 3.

Since there can only be one 3 in a row, the block above the A must be 4 and therefore A = 3.



The combination for 23 is only 6, 8 and 9. 9 cannot be in the top row, therefore B is 9



This block can now only be 2 or 3. 3 is already taken, therefore D is 2

Top right hand block:



The combination for 8 is 1, 2 and 5 (3 has been used). In the top row only 2 and 5 can be used, therefore C is 1.

Final answer: A = 3; B = 9; C = 1;

D = 2 and E = 3.

There is no need to complete the whole Sudoku to get to the answer but the full solution is the following:

[8]

16	9	7	4	1	3	6	8	2	5
16	2	8	A	5	4	B	6	7	C
6	1	5	7	8	2	3	4	9	
5	1	E	8	6	2	4	9	5	7
4	9	2	3	7	5	1	6	8	
15	7	5	6	9	1	8	2	3	4
8	6	1	4	5	3	7	9	2	
8	2	4	7	2	9	1	5	8	6
5	2	9	8	6	7	4	1	3	

THE END